Universal Topological Marker in Dirac Model

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Contents

§1 The Clifford Algebra representation of the Orthogonal groups

$\Re$.1 Dirac metrics-the basis of the clifford algebra

in order to define the Dirac type hamiltonian, we try to define a Clifford algebra Cl^n for the group SO(n), namely, the bi-vectors in the Clifford algebra Cl^n form the basis of the Lie algebra of the group SO(n).

since the construction of the Clifford algebra can be very hard for arbitrary dimension, at present we consider the Clifford algebra Cl^{2n+1} for the group SO(2n+1) by consider the construction using conduction, since we know that the Clifford algebra Cl^3 for SO(3) is spanned by

$$\text{Span}_R\{1, \sigma_x, \sigma_y, \sigma_z, \sigma_x\sigma_y, \sigma_x\sigma_z, \sigma_y\sigma_z, \sigma_x\sigma_y\sigma_z\} \cong C^{2\times 2}$$

note that even if $\sigma_x\sigma_y = i\sigma_z$, $\sigma_x\sigma_y$ and σ_z is linear independent over R.

and then define the Cliford algebra Cl^{2n+1} by induction from Cl^{2n-1}

$$\Gamma_{i<2n} = \Gamma'_i \otimes \sigma_x \quad \Gamma_{2n} = I \otimes \sigma_y \quad \Gamma_{2n+1} = I \otimes \sigma_z$$

thus this representation of the Basis of the Clifford algebra Cl^{2n+1} is 2^n dimensional metrics by construction which is also the minimum dimension of metrics representation of the Clifford Algebra Cl^{2n+1}. since the total number of independent metrics with respect to the real R is

$$2 \times 2^n \times 2^n = 2^{2n+1}$$

which is the same as the real dimension of the Clifford algebra Cl^{2n+1},actually this is due to the basic fact that $Cl^{8k+l} \cong \otimes_k R^{16\times 16} \otimes Cl^l$ and $Cl^3 \cong C^{2\times 2}$, $Cl^5 \cong H^{2\times 2} \oplus H^{2\times 2}$ and $Cl^7 \cong C^{8\times 8}$, we know that the behavior of Cl^{2n+1} is similar for even n which is quite different from the odd n.

within the above construction, the pseudo-scalar is just

$$\omega = \Gamma_1\Gamma_2 \cdots \Gamma_{2n+1} = i^n$$

and the metrics Γ_{2k+1} is real and Γ_{2k} is purely imaginary, and we can define product of all the real metrics

$$B = \Gamma_1\Gamma_3 \cdots \Gamma_{2n+1} \to B^{-1} = B^\dagger = \Gamma_{2n+1}\Gamma_{2n-1} \cdots \Gamma_3\Gamma_1$$

and we can compute that

$$B\Gamma_{2k}B^{-1} = \Gamma_1\Gamma_3 \cdots \Gamma_{2n+1}\Gamma_{2k}\Gamma_{2n+1}\Gamma_{2n-1} \cdots \Gamma_3\Gamma_1 = (-1)^{n+1}\Gamma_{2k}BB^{-1} = (-1)^{n+1}\Gamma_{2k} = (-1)^n\Gamma^*_{2k}$$

since we know $\Gamma^*_{2k} = -\Gamma_{2k}$, furthermore

$$\begin{aligned}
B\Gamma_{2k+1}B^{-1} &= \Gamma_1\Gamma_3 \cdots \Gamma_{2n+1}\Gamma_{2k+1}\Gamma_{2n+1}\Gamma_{2n-1} \cdots \Gamma_3\Gamma_1 = (-1)^{n-k}\Gamma_1\Gamma_3 \cdots \Gamma_{2k+1}\Gamma_{2k+1}\Gamma_{2k+3}\Gamma_{2n+1}\Gamma_{2n+1}\Gamma_{2n-1} \cdots \Gamma_3\Gamma_1 \\
&= (-1)^{n-k}\Gamma_1\Gamma_3 \cdots \Gamma_{2k+1}\Gamma_{2k+1}\Gamma_{2k+1} \cdots \Gamma_3\Gamma_1 = (-1)^{n-k}\Gamma_1\Gamma_3 \cdots \Gamma_{2k+1} \cdots \Gamma_3\Gamma_1 \\
&= (-1)^{n-k}(-1)^k\Gamma_{2k+1}\Gamma_1\Gamma_3 \cdots \Gamma_{2k-1}\Gamma_{2k-1} \cdots \Gamma_3\Gamma_1 = (-1)^{n-k}\Gamma_{2k+1}\Gamma_1\Gamma_3 \cdots \cdots \Gamma_3\Gamma_1 \\
&= (-1)^n\Gamma_{2k+1} = (-1)^n\Gamma^*_{2k+1}
\end{aligned}$$

since Γ_{2k+1} is real metrics, so we can write the above into a compact form as

$$B\Gamma_iB^{-1} = (-1)^n\Gamma^*_i$$

we need this operator B and the above equation due to the fact that we want to construct anti-linear symmetry operator time-reversal T and Particle-Hole C from B in the following context in considering the Dirac model.

$\Re$.2 The symmetry of the Dirac metrics

there is a degree of freedom of choosing these Dirac metrics, given a basis of Clifford algebra for $SO(2n+1)$:

$$\Gamma_1, \Gamma_2, \cdots, \Gamma_{2n+1}$$

we can choose an elements of SO(2n+1), namely O and consider the following metrics

$$\{\gamma_i = \sum_j O_{i,j}\Gamma_j\}$$

we can directly calculate that

$$\{\gamma_i, \gamma_j\} = O_{i,k}O_{j,l}\{\Gamma_k, \Gamma_l\} = 2O_{i,k}O_{j,l}\delta_{k,l} = 2O_{i,k}O_{j,k} = 2\delta_{i,j}$$

which means that this set of metrics can also be used as the basis. so there is SO(2n+1) degree of freedom of choosing the basis.

besides, we can find that

$$\begin{aligned}
&\gamma_1\gamma_2\gamma_3\cdots\gamma_{2n+1} \\
&= \sum_{i_1,i_2,\cdots,i_{2n+1}} O_{1,i_1}O_{2,i_2}\cdots O_{2n+1,i_{2n+1}}\Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_{2n+1}} \\
&= \sum_{\sigma\in S^{2n+1}} ((i_1, i_2, \cdots, i_{2n+1}) = \sigma(1,2,\cdots,2n+1))O_{1,i_1}O_{2,i_2}\cdots O_{2n+1,i_{2n+1}}\Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_{2n+1}} \\
&= O_{1,i_1}O_{2,i_2}\cdots O_{2n+1,i_{2n+1}}\epsilon_{i_1,i_2,\cdots,i_{2n+1}}\Gamma_1\Gamma_2\cdots\Gamma_{2n+1} \\
&= O_{1,i_1}O_{2,i_2}\cdots O_{2n+1,i_{2n+1}}\epsilon_{i_1,i_2,\cdots,i_{2n+1}}i^n \\
&= \det(O)i^n \\
&= i^n
\end{aligned}$$

in the above, we have used the fact if any two index i_k, i_l is the same, then the sum is zero, for example if in choosing the term from the second one such that $i_2 = i_1$, then

$$\sum_{i_1} O_{1,i_1}O_{2,i_1}\Gamma_{i_1}\Gamma_{i_1}\gamma_3\cdots\gamma_{2n+1} = \sum_{i_1} O_{1,i_1}O_{2,i_1}\gamma_3\cdots\gamma_{2n+1} = 0$$

since O is an element of SO(2n+1). so the non-vanishing term after summation must satisfying $i_2 \neq i_1$ and so on.

§2 The Dirac model for different symmetry class and different spatial dimension

in this section, we try to consider the Dirac type hamiltonian for any symmetry class and any spatial dimension. which can be written in momentum space as:

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$$

and the Γ_i are the elements of some kind of basis of Clifford algebra.

ℜ.1 The hamiltonian constructed for the two complex classes

we start from the two complex class A and AIII which contain no anti-linear operators.

for any spatial dimension D in these two complex classes, we can choose $n = [\frac{D+1}{2}]$, and consider the Clifford algebra Cl^{2n+1}

if D is even, $D = 2m \rightarrow n = m, 2n+1 = 2m+1 = D+1$, the Clifford algebra is Cl^{2m+1} thus we choose $\Gamma_0 = \Gamma_{D+1} = \Gamma_{2m+1}$, and there is no extra Γ left behind for us to define the Chiral symmetry operator(which anti-commute with $H(\mathbf{k})$), since we have constructed the hamiltonian in D dimensional as $H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$,thus this hamiltonian belongs to the symmetry class A.

$$A: \quad D = 2m \quad H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i \quad \Gamma_0 = \Gamma_{D+1} = \Gamma_{2m+1} \quad Cl^{2m+1}$$

if D is odd, $D = 2m-1 \rightarrow n = m, 2n+1 = 2m+1$, the Clifford algebra is Cl^{2m+1} thus we can choose $\Gamma_0 = \Gamma_{D+1} = \Gamma_{2m}$, and there is also a Γ left behind which is $\Gamma_{D+2} = \Gamma_{2m+1}$, since it anti-commute with all the $\Gamma_{i\leq D}$, it can serve as an chiral operator for the hamiltonian $H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$, thus this hamiltonian belongs to the class AIII.

$$AIII: \quad D = 2m-1 \quad H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i \quad \Gamma_0 = \Gamma_{D+1} = \Gamma_{2m}, S = \Gamma_{2m+1} \quad Cl^{2m+1}$$

so we have constructed the required hamiltonian for these two complex symmetry classes which can take non-trivial topological order.

in the dimension $D = 2m-1$, if the system takes no chiral symmetry, then we can add extra term $r_{D+1}\Gamma_{2m+1}$ to the hamiltonian

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i + r_{D+1}\Gamma_{2m+1}$$

then the classification of this hamiltonian can be characteried by the homotopy group $\pi_{2m-1}(S^{2m})$, which is trivial, this corresponds to the case that in odd dimension the symmetry class A takes trivial topological phases.

ℜ.2 the hamiltonian constructed for the eight real classes

in the real class, there is anti-linear operators which make a connection between $H(k)$ and $H(-k)$, so in this case, we impose extra symmetry on the choosing of the parameters for the Dirac type hamiltonian, since in the gap closing point, the energy band behaves like $\vec{k}\cdot\vec{\Gamma}$, in the hamiltonian

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$$

we impose(which is essentially the case for the linear Dirac Model)

$$r_i(\mathbf{k}) = -r_i(-\mathbf{k}) \quad r_0(\mathbf{k}) = r_0(-\mathbf{k})$$

which means that r_0 behaves as the mass term in the model and r_i behaves as the momentum. and we have

$$TH(\mathbf{k})^*T^{-1} = H(-\mathbf{k}) \rightarrow r_0(\mathbf{k})T\Gamma_0^*T^{-1} = r_0(-\mathbf{k})\Gamma_0, r_i(\mathbf{k})T\Gamma_i^*T^{-1} = r_i(-\mathbf{k})\Gamma_i$$

which means that

$$T\Gamma_0^* T^{-1} = \Gamma_0 \quad T\Gamma_i^* T^{-1} = -\Gamma_i$$

since we have imposed the constrain on the parameters

similarly, for the particle-hole symmetry,

$$CH(\mathbf{k})^* C^{-1} = -H(-\mathbf{k}) \to r_0(\mathbf{k}) C\Gamma_0^* C^{-1} = -r_0(-\mathbf{k})\Gamma_0, r_i(\mathbf{k}) C\Gamma_i^* C^{-1} = -r_i(-\mathbf{k})\Gamma_i$$

which means that

$$C\Gamma_0^* C^{-1} = -\Gamma_0 \quad C\Gamma_i^* C^{-1} = \Gamma_i$$

∮.1 The Primary Series

in order to discuss all the real cases, we at first try to consider the diagonal entry of the periodic table[3] which is called the primary series in literature. for the spatial dimension D, we consider $n = [\frac{D+1}{2}]$ and set $\Gamma_0 = \Gamma_{D+1}$ as before.

Even D case: when D is even, $D = 2m \to n = m, 2n+1 = 2m+1, D+1 = 2m+1, \Gamma_0 = \Gamma_{D+1} = \Gamma_{2m+1}$,the Cliffod Algebra is Cl^{2m+1}, there is no extra Γ_i left behind and there is also no chiral symmetry in this case. what left for us is to find the possible T or C operators. define

$$A = B\Gamma_0$$

we can find that

$$\begin{aligned} A\Gamma_i^* &= B\Gamma_0\Gamma_i^* = \Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m+1}\Gamma_i^* = (-1)^m\Gamma_{2m+1}\Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_i^* = (-1)^m\Gamma_0 B\Gamma_i^* \\ &= (-1)^m\Gamma_0(-1)^m\Gamma_i B = (-1)^{1+m}\Gamma_i B\Gamma_0 = (-1)^{m+1}\Gamma_i A \\ \to A\Gamma_i^* A^{-1} &= (-1)^{m+1}\Gamma_i \end{aligned}$$

similarly, we have

$$\begin{aligned} A\Gamma_0^* &= B\Gamma_0\Gamma_0^* = \Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m+1}\Gamma_{2m+1}^* = (-1)^m\Gamma_{2m+1}\Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_0^* = (-1)^m\Gamma_0 B\Gamma_0^* \\ &= (-1)^m\Gamma_0(-1)^m\Gamma_0 B = (-1)^m\Gamma_0 B\Gamma_0 = (-1)^m\Gamma_0 A \\ \to A\Gamma_0^* A^{-1} &= (-1)^m\Gamma_0 \end{aligned}$$

so if m is even, A behaves like Time reversal symmetry and if m is odd, A behaves Particle-Hole symmetry on the hamiltonian

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$$

besides, we have

$$AA^* = B\Gamma_0 B\Gamma_0 = \Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m+1}\Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m+1} = (-1)^{\frac{(m-1)m}{2}}$$

thus we can conclude which symmetry class the hamiltonian belongs to, namely

$$\begin{aligned} &D = 2, m = 1, A = C, CC^* = +1, \text{symmetry class } D \\ &D = 4, m = 2, A = T, TT^* = -1, \text{symmetry class } AII \\ &D = 6, m = 3, A = C, CC^* = -1, \text{symmetry class } C \\ &D = 8, m = 4, A = T, TT^* = +1, \text{symmetry class } AI \end{aligned}$$

Odd D case: when D is odd, $D = 2m-1 \rightarrow n = m, 2n+1 = 2m+1, D+1 = 2m, \Gamma_0 = \Gamma_{D+1} = \Gamma_{2m}$,the Cliffod Algebra is Cl^{2m+1}, there is an extra Γ_{2m+1} left behind and there is chiral symmetry in this case, we can define it as $S = \Gamma_{D+2} = \Gamma_{2m+1}$ since we have omit it in the hamiltonian. what left for us is to find the possible T and C operators. define

$$A = B\Gamma_0$$

we can find that

$$\begin{aligned} A\Gamma_i^* &= B\Gamma_0\Gamma_i^* = \Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m}\Gamma_i^* = (-1)^{m+1}\Gamma_{2m}\Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_i^* = (-1)^{m+1}\Gamma_0 B\Gamma_i^* \\ &= (-1)^{m+1}\Gamma_0(-1)^m\Gamma_i B = -\Gamma_0\Gamma_i B = (-1)^{1+1+m+1}\Gamma_i B\Gamma_0 = (-1)^{m+1}\Gamma_i A \\ \rightarrow A\Gamma_i^* A^{-1} &= (-1)^{m+1}\Gamma_i \end{aligned}$$

similarly, we have

$$\begin{aligned} A\Gamma_0^* &= B\Gamma_0\Gamma_0^* = \Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m}\Gamma_{2m}^* = (-1)^{m+1}\Gamma_{2m}\Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m}^* = (-1)^{m+1}\Gamma_0 B\Gamma_0^* \\ &= (-1)^{m+1}\Gamma_0(-1)^m\Gamma_0 B = -(-1)^{m+1}\Gamma_0 B\Gamma_0 = (-1)^m\Gamma_0 A \\ \rightarrow A\Gamma_0^* A^{-1} &= (-1)^m\Gamma_0 \end{aligned}$$

so if m is even, A behaves like Time reversal symmetry and if m is odd, A behaves Particle-Hole symmetry on the hamiltonian, which is the same as the even case,

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$$

besides, we have

$$AA^* = B\Gamma_0 B(-\Gamma_0) = -\Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m}\Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m} = (-1)^{1+m+1+\frac{m(m+1)}{2}} = (-1)^{\frac{m(m-1)}{2}}$$

which is also the same the the even case, besides, since AS is another anti-linear operator which is a different type of A and it squares to

$$(AS)(AS)^* = (-1)^{\frac{m(m-1)}{2}+m+1} = -(-1)^{\frac{m(m+1)}{2}}$$

so we can directly derive the symmetry class that the hamiltonian belongs to, namely

$$\begin{aligned} &D = 1, m = 1, A = C, CC^* = +1, AS = T, TT^* = +1, \text{symmetry class } BDI \\ &D = 3, m = 2, A = T, TT^* = -1, AS = C, CC^* = +1, \text{symmetry class } DIII \\ &D = 5, m = 3, A = C, CC^* = -1, AS = T, TT^* = -1, \text{symmetry class } CII \\ &D = 7, m = 4, A = T, TT^* = +1, AS = C, CC^* = -1, \text{symmetry class } CI \end{aligned}$$

thus we can find that when D goes from 0 to 7 the Dirac type hamiltonian constructed with the help of the Clifford algebra $Cl^{[\frac{D+1}{2}]}$

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$$

goes around the eight symmetry classes, respectively. Thus we have work out all the diagonal symmetry class in the specific spatial dimension D.

∮.2 The Even Series

in this part, we construct the Dirac type Hamiltonian for the even series which is labeled by the topological number 2Z. In this case, for the spatial dimension D, we consider $n = [\frac{D+3}{2}]$, which give us extra two Dirac Gamma Metrics unused in the construction of the hamiltonian and we define

$$\Gamma_0 = -i\Gamma_{D+1}\Gamma_{D+2}\Gamma_{D+3}$$

Even D case: when D is even, $D = 2m \to n = m+1, 2n+1 = 2m+3, D+1 = 2m+1, \Gamma_0 = -i\Gamma_{2m+1}\Gamma_{2m+2}\Gamma_{2m+3}$,the Cliffod Algebra is Cl^{2m+3}, there is no extra Γ_i left behind and there is also no chiral symmetry in this case. what left for us is to find the possible T or C operators. we can find that

$$\begin{aligned} B\Gamma_i^* &= (-1)^n\Gamma_i B = (-1)^{m+1}\Gamma_i B \\ \to B\Gamma_i^* B^{-1} &= (-1)^{m+1}\Gamma_i \end{aligned}$$

similarly, we have

$$\begin{aligned} B\Gamma_0^* &= B(-i\Gamma_{2m+1}\Gamma_{2m+2}\Gamma_{2m+3})^* = iB\Gamma_{2m+1}^*\Gamma_{2m+2}^*\Gamma_{2m+3}^* = i(-1)^{3(m+1)}\Gamma_{2m+1}\Gamma_{2m+2}\Gamma_{2m+3}B \\ &= (-1)^m - i\Gamma_{2m+1}\Gamma_{2m+2}\Gamma_{2m+3}B = (-1)^m\Gamma_0 B \\ \to B\Gamma_0^*B^{-1} &= (-1)^m\Gamma_0 \end{aligned}$$

so if m is even, B act as the Time reversal operator and if m is odd, B act as the Particle hole operator on the hamiltonian

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$$

besides, we have

$$BB^* = \Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m+3}\Gamma_1\Gamma_3\cdots\Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m+3} = (-1)^{\frac{(m+1)(m+2)}{2}}$$

thus we can conclude which symmetry class the hamiltonian belongs to, namely

$$\begin{aligned} &D = 2, m = 1, B = C, CC^* = -1, \text{symmetry class } C \\ &D = 4, m = 2, B = T, TT^* = +1, \text{symmetry class } AI \\ &D = 6, m = 3, B = C, CC^* = +1, \text{symmetry class } D \\ &D = 8, m = 4, B = T, TT^* = -1, \text{symmetry class } AII \end{aligned}$$

Odd D case: when D is odd, $D = 2m-1 \to n = m+1, 2n+1 = 2m+3, D+1 = 2m, \Gamma_0 = -i\Gamma_{2m}\Gamma_{2m+1}\Gamma_{2m+2}$,the Cliffod Algebra is Cl^{2m+3}, there is an extra $\Gamma_{2m+3} = \Gamma_{D+4}$ left behind and there is also chiral symmetry in this case, so we can define S=$\Gamma_{D+4} = \Gamma_{2m+3}$. what left for us is to find the possible T and C operators. we can find that

$$\begin{aligned} B\Gamma_i^* &= (-1)^n\Gamma_i B = (-1)^{m+1}\Gamma_i B \\ \to B\Gamma_i^* B^{-1} &= (-1)^{m+1}\Gamma_i \end{aligned}$$

similarly, we have

$$\begin{aligned} B\Gamma_0^* &= B(-i\Gamma_{2m}\Gamma_{2m+1}\Gamma_{2m+2})^* = iB\Gamma_{2m}^*\Gamma_{2m+1}^*\Gamma_{2m+2}^* = i(-1)^{3(m+1)}\Gamma_{2m}\Gamma_{2m+1}\Gamma_{2m+2}B \\ &= (-1)^m - i\Gamma_{2m}\Gamma_{2m+1}\Gamma_{2m+2}B = (-1)^m\Gamma_0 B \end{aligned}$$

$$\rightarrow B\Gamma_0^* B^{-1} = (-1)^m \Gamma_0$$

so if m is even, B act as the Time reversal operator and if m is odd, B act as the Particle hole operator on the hamiltonian as the even case

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$$

besides, we have

$$BB^* = \Gamma_1\Gamma_3 \cdots \Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m+3}\Gamma_1\Gamma_3 \cdots \Gamma_{2m-1}\Gamma_{2m+1}\Gamma_{2m+3} = (-1)^{\frac{(m+1)(m+2)}{2}}$$

and BS is another type of anti-linear operator which is different from B, and BS satisfying

$$(BS)(BS)^* = (-1)^{m+1+\frac{(m+1)(m+2)}{2}} = (-1)^{\frac{m(m+1)}{2}}$$

thus we can conclude which symmetry class the hamiltonian belongs to, namely

$$D = 1, m = 1, B = C, CC^* = -1, BS = T, TT^* = -1, \text{symmetry class } CII$$
$$D = 3, m = 2, B = T, TT^* = +1, BS = C, CC^* = -1, \text{symmetry class } CI$$
$$D = 5, m = 3, B = C, CC^* = +1, BS = T, TT^* = +1, \text{symmetry class } BDI$$
$$D = 7, m = 4, B = T, TT^* = -1, BS = C, CC^* = +1, \text{symmetry class } DIII$$

thus we can find that when D goes from 0 to 7 the Dirac type hamiltonian constructed with the help of the Clifford algebra $Cl^{[\frac{D+3}{2}]}$

$$H(\mathbf{k}) = \sum_{i=0}^{D} r_i(\mathbf{k})\Gamma_i$$

goes around the eight symmetry classes, respectively. Thus we have work out all the even series in the specific spatial dimension D.

∮.3 The first and second descendants

these two cases are simply obtained from the primary series with the same symmetry class by going one or two dimensional lower by setting $r_D = 0(r_D = 0, r_{D-1} = 0)$, respectively.

in the primary series, we find that for odd D

$$\Gamma_0\Gamma_1 \cdot \Gamma_D S = (-1)^D \Gamma_1 \cdot \Gamma_D \Gamma_{D+1}\Gamma_{D+2} = (-1)^D i^{[\frac{D+1}{2}]} = -i^{\frac{D+1}{2}}$$

in the even series, we find that for odd D

$$\Gamma_0\Gamma_1 \cdot \Gamma_D S = -i\Gamma_{D+1}\Gamma_{D+2}\Gamma_{D+3}\Gamma_1 \cdot \Gamma_D\Gamma_{D+4} = -i(-1)^D \Gamma_1 \cdot \Gamma_D\Gamma_{D+1}\Gamma_{D+2}\Gamma_{D+3}\Gamma_{D+4} = i \cdot i^{\frac{D+3}{2}} = -i^{\frac{D+1}{2}}$$

in both case, for odd D, we have

$$\Gamma_0\Gamma_1 \cdot \Gamma_D S = -i^{\frac{D+1}{2}}$$

similarly, in the primary series, for the even D, we have

$$\Gamma_0\Gamma_1 \cdot \Gamma_D = (-1)^D \Gamma_1 \cdot \Gamma_D\Gamma_{D+1} = i^{\frac{D}{2}}$$

in the even series, for the even D, we have

$$\Gamma_0\Gamma_1 \cdot \Gamma_D = -i\Gamma_{D+1}\Gamma_{D+2}\Gamma_{D+3}\Gamma_1 \cdot \Gamma_D = -i(-1)^D \Gamma_1 \cdot \Gamma_D\Gamma_{D+1}\Gamma_{D+2}\Gamma_{D+3} = -ii^{\frac{D+2}{2}} = i^{\frac{D}{2}}$$

so, in both case, for even D, we have

$$\Gamma_0\Gamma_1 \cdot \Gamma_D = i^{\frac{D}{2}}$$

all the above discussion are summarized in the Figure(1)

TABLE II. Pairs $(T, C)_n$ for the primary series (diagonal) and the even series (remaining entries). The subindex n indicates the size of the Dirac matrices which are 2^n dimensional. Blue entries satisfy $XX^* = +1$ and red entries satisfy $XX^* = -1$, where $X = T, C$. Here, B is the product of all real Γ_i, $A = B\Gamma_0$ (where Γ_0 is the Γ_i that plays the role of the "mass term" in the Dirac Lagrangian), and S is the Γ_i that implements the chiral symmetry. The arrows indicate the descendants that simply inherit the choices from the primary series. The table continues periodically for $D > 7$, with the index n increasing by 4 with each period.

	$D=0$	$D=1$	$D=2$	$D=3$	$D=4$	$D=5$	$D=6$	$D=7$
AI	$(A,-)_0$				$(B,-)_3$		$\leftarrow$	$\leftarrow$
BDI	$\leftarrow$	$(AS,A)_1$				$(BS,B)_4$		$\leftarrow$
D	$\leftarrow$	$\leftarrow$	$(-,A)_1$				$(-,B)_4$	
DIII		$\leftarrow$	$\leftarrow$	$(A,AS)_2$				$(B,BS)_5$
AII	$(B,-)_1$		$\leftarrow$	$\leftarrow$	$(A,-)_2$			
CII		$(BS,B)_2$		$\leftarrow$	$\leftarrow$	$(AS,A)_3$		
C			$(-,B)_2$		$\leftarrow$	$\leftarrow$	$(-,A)_3$	
CI				$(B,BS)_3$		$\leftarrow$	$\leftarrow$	$(A,AS)_4$

Figure 1: the operators in different symmetry class and spatial dimensions

$\Re$.3 Some comments on the above construction

in the following context, we refer to the specific choice of Dirac Metrics Γ_i to be the one constructed in section (1).

in the above proof, it greatly relies on the specific construction of the Dirac metrics and the order of these metrics. since there is SO(2n+1) degree of choosing the Dirac metrics (the basis) and S^{2n+1} symmetry(the order of these metrics), in this section, let us make some comments on the above strategy.

- there is $O(n+1) \times O(n) \subset O(2n+1)$ degree of freedom of choosing these basis so as the above strategy still works.

  the metrics B defined above is $B = \Gamma_1\Gamma_3\Gamma_5 \cdots \Gamma_{2n+1}$, which can be extended to the product of all the real metrics but there is a restriction to the basis choice that it is either real or purely imaginary.

  consider a basis transformation $\gamma_i = O_{i,j}\Gamma_j$, then $\gamma_i^* = O_{i,j}\Gamma_j^*$, since $\Gamma_{2m+1}^* = \Gamma_{2m+1}$ and $\Gamma_{2m}^* = -\Gamma_{2m}$, so if γ_i is real, it requires that $O_{i,2k} = 0$, if γ_i is purely imaginary, it requires that $O_{i,2k+1} = 0$, so in order to derive n+1 real basis and n purely imaginary basis, the choice of O is reduced from $O(2n+1)$ to $O(n+1) \times O(n)$, in this sense, B is well defined as the product of all the real metrics and it satisfying

  $$B\gamma_i B^{-1} = (-1)^n \gamma_i^*$$

  since if γ_i is real, then it should be exchange n times from the right of B to the left of B which will give us $(-1)^n \gamma_i BB^{-1} = (-1)^n \gamma_i^*$ due to the construction of B.

  if γ_i is purely imaginary, then it should be exchange n+1 times from the right of B to the left of B which will give us $(-1)^{n+1}\gamma_i BB^{-1} = (-1)^{n+1}(-\gamma_i^*) = (-1)^n \gamma_i^*$

  Obviously, not all the transformation will give us n+1 real basis and n purely imaginary basis, for example, $\gamma_1 = \sigma_x, \gamma_2 = -\sin\theta\sigma_3 + \cos\theta\sigma_2, \gamma_3 = \cos\theta\sigma_3 + \sin\theta\sigma_2$ is also a basis of the Clifford algebra of Cl^3, but there is no purely imaginary metrics.

- the ordering of these Dirac metrics has no influence in the above strategy.

  this is to say, the choice of γ_0 can be arbitrary.define $A = B\gamma_0$ with arbitrary γ_0, we can fallow the above proof with this extended B.

As for the primary series, in the Even D case, $D = 2m \to n = m$,as for any $\gamma_i \neq \gamma_0$, we have

$$\begin{aligned}
&A\gamma_i^* = B\gamma_0\gamma_i^* = (-1)^{(n+1)/n}\gamma_0 B\gamma_i^* = (-1)^n(-1)^{(n+1)/n}\gamma_0\gamma_i B \\
&= (-1)(-1)^n(-1)^{(n+1)/n}\gamma_i\gamma_0 B = (-1)(-1)^n(-1)^{(n+1)/n}(-1)^{(n+1)/n}\gamma_i\gamma_0 B = (-1)^{n+1}\gamma_i A \\
&\to A\gamma_i^* A^{-1} = (-1)^{n+1}\gamma_i
\end{aligned}$$

but for γ_0, we have that

$$\begin{aligned}
&A\gamma_0^* = B\gamma_0\gamma_0^* = (-1)^{(n+1)/n}\gamma_0 B\gamma_i^* = (-1)^n(-1)^{(n+1)/n}\gamma_0\gamma_0 B \\
&= (-1)^n(-1)^{(n+1)/n}\gamma_0\gamma_0 B = (-1)^n(-1)^{(n+1)/n}(-1)^{(n+1)/n}\gamma_0\gamma_0 B = (-1)^n\gamma_0 A \\
&\to A\gamma_0^* A^{-1} = (-1)^n\gamma_0
\end{aligned}$$

besides, if γ_0 is chosen to be the purely imaginary:

$$AA^* = B\gamma_0 B\gamma_0^* = B\gamma_0 B(-\gamma_0) = B\gamma_0(-\gamma_0)(-1)^{n+1}B = (-1)^n B^2$$

if γ_0 is chosen to be the real one:

$$AA^* = B\gamma_0 B\gamma_0^* = B\gamma_0 B(\gamma_0) = B\gamma_0\gamma_0(-1)^n B = (-1)^n B^2$$

in both cases, the result is the same

$$AA^* = (-1)^n B^2 = (-1)^{n+\frac{n(n+1)}{2}} = (-1)^{\frac{n(n-1)}{2}}$$

which is the same as the previous result which we take the specific form of Γ_i and ordering.

for the Odd D case in the primary series, $D = 2m - 1 \to n = m$, which is the same as the previous one since the Clifford algebra we considering is the same Cl^{2n+1}.

but in this case we can choose another arbitrary one as our chiral operator $S \neq \gamma_0$, then we have

$$(AS)(AS)^* = ASA^*S^* = (-1)^{n+1}ASSA^* = (-1)^{n+1}(-1)^{\frac{n(n-1)}{2}} = -(-1)^{\frac{n(n+1)}{2}}$$

regardless of real or purely imaginary choice of S. which is the same as the above specific choice of Γ_i and ordering.

as for the even series, for the even D=2m, n=m+1, and in this case, we can choose arbitrary $\gamma_0 = -i\gamma_i\gamma_j\gamma_k$ and find that

$$B\gamma_i^* = (-1)^n\gamma_i B = (-1)^{m+1}\gamma_i B \to B\gamma_i^* B^{-1} = (-1)^{m+1}\gamma_i$$

as for γ_0, we have

$$\begin{aligned}
&B\gamma_0^* = iB\gamma_i^*\gamma_j^*\gamma_k^* = i(-1)^n\gamma_i\gamma_j\gamma_k B = (-1)^{n-1}\gamma_0 B = (-1)^m\gamma_0 B \\
&\to B\gamma_0^* B^{-1} = (-1)^m\gamma_0
\end{aligned}$$

besides, we have

$$BB^* = B^2 = (-1)^{\frac{n(n+1)}{2}} = (-1)^{\frac{m(m+1)}{2}}$$

which is the same as the previous result with the specific form of Γ_i and ordering.

for the odd D=2m-1, n=m+1, in this case, after choosing arbitrary $\gamma_0 = -i\gamma_i\gamma_j\gamma_k$, we can choose arbitrary chiral operator $S \neq \gamma_{i,j,k}$, then we have

$$(BS)(BS)^* = BSB^*S^* = (-1)^n BSSB = (-1)^n B^2 = (-1)^n(-1)^{\frac{n(n+1)}{2}} = (-1)^{\frac{n(n-1)}{2}} = (-1)^{\frac{m(m+1)}{2}}$$

which is also the same as the previous result.

we need to notice that in this case, the choice of γ_0 can also be extended to $\gamma_0 = -i\gamma_j$ since

$$B\gamma_0^* = iB\gamma_j^* = i(-1)^n\gamma_j B = (-1)^{n-1}\gamma_0 B \rightarrow B\gamma_0^* B^{-1} = (-1)^{n-1}\gamma_0$$

which means that it can also be used as the mass term generator!

In conclusion, the choice of γ_0 and S can be arbitrary in this strategy as long as there is always n+1 real basis and n purely imaginary basis. In other words, there is $O(n+1) \times O(n)$ degree of freedom for choosing the basis, under this constrain, the ordering of these metrics does not affect the whole strategy.

there is also one thing to be clarified, that is we should ordering the other Dirac metrics to meet the requirement that.
for even D

$$\Gamma_0\Gamma_1 \cdot \Gamma_D = i^{\frac{D}{2}}$$

for odd D

$$\Gamma_0\Gamma_1 \cdot \Gamma_D S = -i^{\frac{D+1}{2}}$$

§3 Unified Topological Invariants: The Wrapping Number[1]

ℜ.1 The degree of a map

we can consider two manifolds N and M of the same dimension D, and a map $f : N \rightarrow M$, lLet M be orientable with a volume form ω that is nowhere vanishing, and define

$$V_M \equiv \int_M \omega$$

the integer-valued degree of the map f is defined as

$$\text{deg} f = \frac{1}{V_M}\int_N f^*\omega$$

where $f^*\omega$ is the pullback of the form ω by the map f, which is also a D-form on N, and this is equivalent to the algebraic definition of the degree which read as

$$\text{deg} f = \sum_{k\in f^{-1}(x_0)} \text{sign} J(k)$$

where J is the Jacobian of the map f

$$J_f(k) = \det \frac{\partial f^\mu}{\partial k^\nu}$$

if M is the sphere $M = S^D$, then we have the above formula by considering S^D embedding in the R^{D+1} and the volume form in S^D is induced from the volume form in R^{D+1} which is trivially defined as $\omega_{R^{D+1}} = dr^0 \wedge dr^1 \wedge \cdots \wedge dr^D$, since the volume form in the manifold with Riemann measure g(x) is $\omega = \sqrt{\text{det} g(x)} d^n x$, and in R^{D+1}, the metrics is just $g(x) = I_n$ since it's Euclidean. since the metric tensor in R^{D+1} is $g = \sum_{i=0}^{D} dx_i dx_i$, so the pullback of this metric tensor to S^D by the map h(y)(where y is the local coordinate in S^D) is ($x_i = h_i(y)$)

$$h^*(g) = \sum_{i=0}^{D} dh_i dh_i = \sum_{i=0}^{D} \frac{\partial h_i(y)}{\partial y_j} dy_j \frac{\partial h_i(y)}{\partial y_k} dy_k = \sum_{j,k} (\sum_{i=0}^{D} \frac{\partial h_i(y)}{\partial y_j} \frac{\partial h_i(y)}{\partial y_k}) dy_j dy_k$$

which mean that the Riemann metric in S^D is

$$g^*_{j,k} = \sum_{i=0}^{D} \frac{\partial h_i(y)}{\partial y_j} \frac{\partial h_i(y)}{\partial y_k} = \frac{\partial h(y)}{\partial y_j} \cdot \frac{\partial h(y)}{\partial y_k}$$

if we choose the local coordinate in S^D as $(n^1, n^2, \cdots, n^D)$ under the map h it maps to $(n^0, n^1, \cdots n^D)$, so we have

$$\frac{\partial h(y)}{\partial y_j} = (\frac{\partial n^0}{\partial n^j}, 0, 0, 0, \cdots, 1, \cdots, 0, 0) \to \frac{\partial h(y)}{\partial y_i} \cdot \frac{\partial h(y)}{\partial y_j} = \frac{\partial n^0}{\partial n^i} \frac{\partial n^0}{\partial n^j} + \delta_{i,j}$$

on the other hand we have

$$\frac{\partial n^0}{\partial n^i} = \frac{\partial \sqrt{1 - \sum_{i=1}^{D} n^{i,2}}}{\partial n^i} = -\frac{n^i}{n^0} \to$$

then we can find that

$$\det(g^*) = \det \begin{pmatrix} 1 & 0 \\ 0 & g^* \end{pmatrix}$$

and we can find that

$$\begin{pmatrix} 1 & 0 \\ 0 & g^* \end{pmatrix} = \begin{pmatrix} n^0 & n^1 & n^2 & \cdots & n^D \\ \frac{\partial n^0}{\partial n^1} & 1 & 0 & \cdots & 0 \\ \frac{\partial n^0}{\partial n^2} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{\partial n^0}{\partial n^D} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} n^0 & n^1 & n^2 & \cdots & n^D \\ \frac{\partial n^0}{\partial n^1} & 1 & 0 & \cdots & 0 \\ \frac{\partial n^0}{\partial n^2} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{\partial n^0}{\partial n^D} & 0 & 0 & \cdots & 1 \end{pmatrix}^T = NN^T$$

where the first row of N is $(n^0, n^1, \cdots, n^D)$ and the (i+1)'s row of N is just $\partial_{n^i}(n^0, n^1, \cdots, n^D)$, so the (i+1,j+1) elements of NN^T is just $\partial_{n^i}(n^0, n^1, \cdots, n^D) \cdot \partial_{n^j}(n^0, n^1, \cdots, n^D) = g^*_{i,j}$, together with the fact that

$$\sum_{i=0}^{D} n_i^2 = 1 \quad n^0 \frac{\partial n^0}{\partial n^j} + n^j = -n^0 \frac{n^j}{n^0} + n^j = 0$$

which make sure the fist row of NN^T is just $(1, 0, 0, \cdots, 0)$, so we have that

$$\sqrt{\det(g^*)} = \det(N) = \det(N^T) = \det(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial n_1}, \frac{\partial \mathbf{n}}{\partial n_2}, \cdots, \frac{\partial \mathbf{n}}{\partial n_D})$$

thus the volume form in S^D can be expressed as

$$\begin{aligned} \omega &= \det(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial n_1}, \frac{\partial \mathbf{n}}{\partial n_2}, \cdots, \frac{\partial \mathbf{n}}{\partial n_D}) dn^1 \wedge dn^2 \wedge \cdots \wedge dn^D \\ &= \epsilon_{i_0 i_1 \cdots i_D} n^{i_0} \frac{\partial n^{i_1}}{\partial n_1} \frac{\partial n^{i_2}}{\partial n_2} \cdots \frac{\partial n^{i_D}}{\partial n_D} dn^1 \wedge dn^2 \wedge \cdots \wedge dn^D \\ &= \epsilon_{i_0 i_1 \cdots i_D} n^{i_0} \frac{\partial n^{i_1}}{\partial n_1} \frac{\partial n^{i_2}}{\partial n_2} \cdots \frac{\partial n^{i_D}}{\partial n_D} \frac{1}{D!} \epsilon_{j_1, j_2 \cdots, j_D} dn^{j_1} \wedge dn^{j_2} \wedge \cdots \wedge dn^{j_D} \\ &= \frac{1}{D!} \det(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial n_1}, \frac{\partial \mathbf{n}}{\partial n_2}, \cdots, \frac{\partial \mathbf{n}}{\partial n_D}) \epsilon_{j_1, j_2 \cdots, j_D} dn^{j_1} \wedge dn^{j_2} \wedge \cdots \wedge dn^{j_D} \\ &= \frac{1}{D!} \det(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial n_{j_1}}, \frac{\partial \mathbf{n}}{\partial n_{j_2}}, \cdots, \frac{\partial \mathbf{n}}{\partial n_{j_D}}) dn^{j_1} \wedge dn^{j_2} \wedge \cdots \wedge dn^{j_D} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{D!}\epsilon_{i_0 i_1 \cdots i_D} n^{i_0} \frac{\partial n^{i_1}}{\partial n_{j_1}} \frac{\partial n^{i_2}}{\partial n_{j_2}} \cdots \frac{\partial n^{i_D}}{\partial n_{j_D}} dn^{j_1} \wedge dn^{j_2} \wedge \cdots \wedge dn^{j_D} \\
&= \frac{1}{D!}\epsilon_{i_0 i_1 \cdots i_D} n^{i_0} dn^{i_1} \wedge dn^{i_2} \wedge \cdots \wedge dn^{i_D}
\end{aligned}
$$

we express it in this form for the reason that it's coordinate independent so as to easily derive the pullback of ω by f to the volume tensor in N, that is

$$
f^*\omega = \frac{1}{D!}\epsilon_{i_0 i_1 \cdots i_D} n^{i_0} dn^{i_1} \wedge dn^{i_2} \wedge \cdots \wedge dn^{i_D} = \det(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial k_1}, \frac{\partial \mathbf{n}}{\partial k_2}, \cdots, \frac{\partial \mathbf{n}}{\partial k_D}) dk^1 \wedge dk^2 \wedge \cdots \wedge dk^D
$$

on the other hand, we know $\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}$, so we have

$$
\begin{aligned}
\det(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial k_1}, \frac{\partial \mathbf{n}}{\partial k_2}, \cdots, \frac{\partial \mathbf{n}}{\partial k_D}) &= \det(\frac{\mathbf{r}}{|\mathbf{r}|}, \frac{1}{|\mathbf{r}|}\frac{\partial \mathbf{r}}{\partial k_1} + \mathbf{r}\frac{\partial \frac{1}{|\mathbf{r}|}}{\partial k_1}, \frac{1}{|\mathbf{r}|}\frac{\partial \mathbf{r}}{\partial k_2} + \mathbf{r}\frac{\partial \frac{1}{|\mathbf{r}|}}{\partial k_2}, \cdots, \frac{1}{|\mathbf{r}|}\frac{\partial \mathbf{r}}{\partial k_D} + \mathbf{r}\frac{\partial \frac{1}{|\mathbf{r}|}}{\partial k_D}) \\
&= \frac{1}{|\mathbf{r}|^{D+1}} \det(\mathbf{r}, \frac{\partial \mathbf{r}}{\partial k_1}, \frac{\partial \mathbf{r}}{\partial k_2}, \cdots, \frac{\partial \mathbf{r}}{\partial k_D})
\end{aligned}
$$

where we have use the fact that adding a column to another column doesn't change the value of the determinate. in conclusion, we have the following maps:

$$
\mathrm{BZ} \xrightarrow{f} S^D \overset{\text{natural embedding}}{\rightarrow} R^{D+1}
$$

$$
\mathbf{k} \xrightarrow{f} \mathbf{x} = f(k) \overset{\text{natural embedding}}{\rightarrow} \mathbf{m}(\mathbf{x})
$$

thus we have $\mathbf{n}(\mathbf{k}) = \mathbf{m}(f(\mathbf{k}))$

$$
\det(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial k_1}, \frac{\partial \mathbf{n}}{\partial k_2}, \cdots, \frac{\partial \mathbf{n}}{\partial k_D}) = J_f(\mathbf{k}) \det(\mathbf{m}, \frac{\partial \mathbf{m}}{\partial x_1}, \frac{\partial \mathbf{m}}{\partial x_2}, \cdots, \frac{\partial \mathbf{m}}{\partial x_D})_{x=f(k)}
$$

then we know

$$
\mathrm{sign} J_f(\mathbf{k}) = \mathrm{sign}\ \det(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial k_1}, \frac{\partial \mathbf{n}}{\partial k_2}, \cdots, \frac{\partial \mathbf{n}}{\partial k_D}) = \mathrm{sign}\ \det(\mathbf{r}, \frac{\partial \mathbf{r}}{\partial k_1}, \frac{\partial \mathbf{r}}{\partial k_2}, \cdots, \frac{\partial \mathbf{r}}{\partial k_D})
$$

since $\det(\mathbf{m}, \frac{\partial \mathbf{m}}{\partial x_1}, \frac{\partial \mathbf{m}}{\partial x_2}, \cdots, \frac{\partial \mathbf{m}}{\partial x_D})_{x=f(k)}$ is just $\sqrt{h^* g} = \sqrt{g^*}$ which is positive due to the fact that the metric in R^{D+1} is identity which is positive.

ℜ.2 The degree of the map from BZ to sphere

as for the Dirac model, the hamiltonian write as

$$
H(\mathbf{k}) = \sum_{i=0}^{D} r^i(\mathbf{k}) \Gamma_i
$$

where these Dirac metrics are specified in the above sections for different symmetry classes and different spatial dimension, since the flattened hamiltonian has the same eigen-states as the original one, we can consider the flattened Hamiltonian

$$
Q(\mathbf{k}) = \sum_{i=0}^{D} n^i(\mathbf{k}) \Gamma_i
$$

with $n^i = \frac{r^i}{|\mathbf{r}|}$, thus for each point k in the BZ(T^D), these parameters lie in the sphere S^D,when k goes around the whole BZ, these vectors will wrap around the Sphere S^D, so we can use the map $\mathbf{n}$ to distinguish topologically distinct hamiltonian, the topological classification of this hamiltonian is just the homotopy groups $[BZ, S^D]$, and the topological number is just the degree of such map $\mathbf{n}$, which is called the wrapping number, defined as

$$\begin{aligned}\deg[\mathbf{n}] &= \frac{1}{V_D}\int_{\mathrm{BZ}} \frac{1}{D!}\epsilon_{i_0\cdots i_D} n^{i_0} dn^{i_1}\wedge dn^{i_2}\wedge\cdots\wedge dn^{i_D}\\ &= \frac{1}{V_D}\int_{\mathrm{BZ}} \epsilon_{i_0\cdots i_D} n^{i_0}\partial_1 n^{i_1}\partial_2 n^{i_2}\cdots\partial_D n^{i_D} d^D\mathbf{k}\\ &= \frac{1}{V_D}\int_{\mathrm{BZ}} \epsilon_{i_0\cdots i_D} r^{i_0}\frac{1}{|\mathbf{r}|^{D+1}}\partial_1 r^{i_1}\partial_2 r^{i_2}\cdots\partial_D r^{i_D} d^D\mathbf{k}\end{aligned}$$

where $V_D = \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})}$ is the volume of the D-Dimensional sphere. The above formula counts how many times the map $\mathbf{n}$ wrap around the sphere as k go around the whole BZ.

there is an alternative way of calculating the above formula, we can pick up a fixed point $\mathbf{n}_0$ in the sphere, and find out how many points $\mathbf{k}_i$ in the BZ which is mapped to this point by $\mathbf{n}$, since the map may be wrapping around the sphere through $\mathbf{n}_0$ at $\mathbf{k}_i$ in the normal direction or in the opposite direction, this orientation is captured by the sign of the following Jocobian:

$$J_{\mathbf{n}}(\mathbf{k}_i) = \epsilon_{i_0\cdots i_D} n^{i_0}\partial_1 n^{i_1}\partial_2 n^{i_2}\cdots\partial_D n^{i_D}|_{\mathbf{k}_i}$$

the the times of the map $\mathbf{n}_i$ wrap around the sphere can be calculated as

$$\deg[\mathbf{n}] = \sum_{\mathbf{k}\ \mathtt{with}\ \mathbf{n}(\mathbf{k})=\mathbf{n}_0} \mathrm{sign} J_{\mathbf{n}}(\mathbf{k})$$

if we choose proper $\mathbf{n}_0$ that all the Jocabians are non-vanishing, so that the above sum is discrete and finite.

$\Re$.3 The winding number represented as the wrapping number

fro the chiral symmetric system in the non-trivial complex class in odd dimension or the primary series in the odd dimension, the hamiltonian are topologically classified by the so called winding number, in D dimension, it read as

$$\nu_D = \frac{(-1)^{\frac{D-1}{2}}(\frac{D-1}{2})!}{D!}(\frac{i}{2\pi})^{\frac{D+1}{2}}\int \mathrm{Tr}(q^{-1}dq)^D$$

where q is the off-diagonal part of the flattened hamiltonian $Q = \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}$ when we choose the chiral operator as $\sigma_z \otimes I_n$, we can compute that

$$\begin{aligned}S(QdQ)^D &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & dq \\ dq^{-1} & 0 \end{pmatrix})^D\\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & dq \\ dq^{-1} & 0 \end{pmatrix})^D\\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\begin{pmatrix} qdq^{-1} & 0 \\ 0 & 0 \quad q^{-1}dq \end{pmatrix})^D\\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\begin{pmatrix} -q^{-1}dq & 0 \\ 0 & 0 \quad q^{-1}dq \end{pmatrix})^D\\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} (-1)^D(q^{-1}dq)^D & 0 \\ 0 & 0 \quad (q^{-1}dq)^D \end{pmatrix}\end{aligned}$$

$$= -\begin{pmatrix} (q^{-1}dq)^D & 0 \\ 0 & (q^{-1}dq)^D \end{pmatrix}$$

thus we know $\mathrm{tr}(q^{-1}dq)^D = -\frac{1}{2}\mathrm{tr}(S(QdQ)^D)$, so we can write the winding number as

$$\nu_D = \frac{(-1)^{\frac{D-1}{2}}(\frac{D-1}{2})!}{D!}(\frac{i}{2\pi})^{\frac{D+1}{2}}\int \mathrm{Tr}[-\frac{1}{2}\mathrm{tr}(S(QdQ)^D)] \quad = (-1)^{\frac{D+1}{2}}\frac{(\frac{D-1}{2})!}{2\times D!}(\frac{i}{2\pi})^{\frac{D+1}{2}}\int \mathrm{tr}(S(QdQ)^D)$$

on the other hand, since we know that $(QdQ)^2 = QdQQdQ = -QQdQdQ = -(dQ)^2$, thus we have $(QdQ)^D = QdQ(QdQ)^{2\times\frac{D-1}{2}} = (-1)^{\frac{D-1}{2}}QdQ(dQ)^{D-1} = (-1)^{\frac{D-1}{2}}Q(dQ)^D$ thus we can write the winding number as

$$\begin{aligned}\nu_D &= (-1)^{\frac{D+1}{2}}\frac{(\frac{D-1}{2})!}{2\times D!}(\frac{i}{2\pi})^{\frac{D+1}{2}}\int(-1)^{\frac{D-1}{2}}\mathrm{tr}(SQ(dQ)^D) \\ &= (-1)^{\frac{D+1}{2}}(-1)\frac{1}{D!}\frac{(\frac{D-1}{2})!}{2\pi^{\frac{D+1}{2}}}(-\frac{i}{2})^{\frac{D+1}{2}}\int \mathrm{tr}(SQ(dQ)^D) \\ &= (-1)^{\frac{D+1}{2}}(-1)\frac{1}{V_D D!}(-\frac{i}{2})^{\frac{D+1}{2}}\int \mathrm{tr}(SQ(dQ)^D)\end{aligned}$$

since we know that $Q = \sum_{i=0}^{D} n^i\Gamma_i$,thus we can express the above winding number as

$$\begin{aligned}\mathrm{tr}(SQ(dQ)^D) &= \mathrm{tr}(S\Gamma_{i_0}\Gamma_{i_1}\cdots\Gamma_{i_D})n^{i_0}dn^{i_1}\wedge\cdots\wedge dn^{i_D} \\ &= \mathrm{tr}(S\Gamma_0\Gamma_1\cdots\Gamma_D)\epsilon_{i_0i_1\cdots i_D}n^{i_0}dn^{i_1}\wedge\cdots\wedge dn^{i_D} \\ &= -i^{\frac{D+1}{2}}\mathrm{tr}(\mathcal{I})\epsilon_{i_0i_1\cdots i_D}n^{i_0}dn^{i_1}\wedge\cdots\wedge dn^{i_D}\end{aligned}$$

thus the winding number is represented as

$$\begin{aligned}\nu_D &= (-1)^{\frac{D+1}{2}}(-1)\frac{1}{V_D D!}(-\frac{i}{2})^{\frac{D+1}{2}}\int -i^{\frac{D+1}{2}}\mathrm{tr}(\mathcal{I})\epsilon_{i_0i_1\cdots i_D}n^{i_0}dn^{i_1}\wedge\cdots\wedge dn^{i_D} \\ &= (-1)^{\frac{D+1}{2}}\mathrm{tr}(\mathcal{I})(\frac{1}{2})^{\frac{D+1}{2}}\frac{1}{V_D}\int\frac{1}{D!}\epsilon_{i_0i_1\cdots i_D}n^{i_0}dn^{i_1}\wedge\cdots\wedge dn^{i_D} \\ &= (-1)^{\frac{D+1}{2}}(\frac{1}{2})^{\frac{D+1}{2}}\mathrm{tr}(\mathcal{I})\deg[\mathbf{n}]\end{aligned}$$

we know that the dimension of the Dirac metrics is 2^n, thus $\mathrm{tr}(\mathcal{I}) = 2^n$, for the complex series and the primary series, we know that $n = [\frac{D+1}{2}] = \frac{D+1}{2}$ since D is odd for winding number, so

$$\nu_D = (-1)^{\frac{D+1}{2}}(\frac{1}{2})^{\frac{D+1}{2}}2^{\frac{D+1}{2}}\deg[\mathbf{n}] = (-1)^{\frac{D+1}{2}}\deg[\mathbf{n}] \tag{1}$$

for the even series, we know that $n = [\frac{D+3}{2}] = \frac{D+3}{2}$, so

$$\nu_D = (-1)^{\frac{D+1}{2}}(\frac{1}{2})^{\frac{D+1}{2}}2^{\frac{D+3}{2}}\deg[\mathbf{n}] = (-1)^{\frac{D+1}{2}}2\deg[\mathbf{n}] \tag{2}$$

this is slightly different from the results derived in the original paper[1] by a factor of $(-1)^{\frac{D+1}{2}}$, since this factor is either +1 or -1 and it's a global factor, which will not change the classification of topological distinct phases. so we can clarify here this extra factor can be removed if we slightly change the definition of the topological invariants.

ℜ.4 The Chern number represented as the wrapping number

as for the Chern number, defined through the berry curvature can be represented as [3]

$$\mathrm{Ch}_{n=\frac{D}{2}} = \frac{1}{n!}(\frac{i}{2\pi})^n\int \mathrm{tr}(\mathcal{F}^n)$$

in the following, we prove in the flattened hamiltonian, this can be expressed as

$$\frac{-1}{2^{2n+1}}\frac{1}{n!}(\frac{i}{2\pi})^n\int \mathrm{tr}(Q(dQ)^{2n})$$

which is also proposed in the article[3].

since $\mathcal{A}^{\alpha,\beta}=\langle u^\alpha(\mathbf{k})|d|u^\beta(\mathbf{k})\rangle=\langle u^\alpha(\mathbf{k})|\partial_{k_i}|u^\beta(\mathbf{k})\rangle dk_i$ so we have

$$\begin{aligned}\mathcal{F}^{\alpha,\beta}&=d\mathcal{A}^{\alpha,\beta}+(A\wedge A)^{\alpha,\beta}\\&=\partial_{k_j}(\langle u^\alpha(\mathbf{k})|\partial_{k_i}|u^\beta(\mathbf{k})\rangle)dk_j\wedge dk_i+\langle u^\alpha(\mathbf{k})|\partial_{k_i}|u^\gamma(\mathbf{k})\rangle\langle u^\gamma(\mathbf{k})|\partial_{k_j}|u^\beta(\mathbf{k})\rangle dk_i\wedge dk_j\\&=(\langle\partial_{k_j}u^\alpha(\mathbf{k})|\partial_{k_i}|u^\beta(\mathbf{k})\rangle)dk_j\wedge dk_i+(\langle u^\alpha(\mathbf{k})|\partial_{k_j}\partial_{k_i}|u^\beta(\mathbf{k})\rangle)dk_j\wedge dk_i+\langle u^\alpha(\mathbf{k})|\partial_{k_i}|u^\gamma(\mathbf{k})\rangle\langle u^\gamma(\mathbf{k})|\partial_{k_j}|u^\beta(\mathbf{k})\rangle dk_i\wedge dk_j\\&=(\langle\partial_{k_j}u^\alpha(\mathbf{k})|\partial_{k_i}|u^\beta(\mathbf{k})\rangle)dk_j\wedge dk_i+\langle u^\alpha(\mathbf{k})|\partial_{k_i}|u^\gamma(\mathbf{k})\rangle\langle u^\gamma(\mathbf{k})|\partial_{k_j}|u^\beta(\mathbf{k})\rangle dk_i\wedge dk_j\end{aligned}$$

we can find that

$$\begin{aligned}\mathrm{tr}(\mathcal{F})&=(\langle\partial_{k_j}u^\alpha(\mathbf{k})|\partial_{k_i}|u^\alpha(\mathbf{k})\rangle)dk_j\wedge dk_i+\langle u^\alpha(\mathbf{k})|\partial_{k_i}|u^\gamma(\mathbf{k})\rangle\langle u^\gamma(\mathbf{k})|\partial_{k_j}|u^\alpha(\mathbf{k})\rangle dk_i\wedge dk_j\\&=(\langle\partial_{k_j}u^\alpha(\mathbf{k})|\partial_{k_i}|u^\alpha(\mathbf{k})\rangle)dk_j\wedge dk_i\\&=(\langle\partial_{k_1}u^\alpha(\mathbf{k})|\partial_{k_2}|u^\alpha(\mathbf{k})\rangle-\langle\partial_{k_2}u^\alpha(\mathbf{k})|\partial_{k_1}|u^\alpha(\mathbf{k})\rangle)dk_1\wedge dk_2\end{aligned}$$

since the second term is vanishing due to the fact that $\langle u^\alpha(\mathbf{k})|\partial_{k_i}|u^\gamma(\mathbf{k})\rangle\langle u^\gamma(\mathbf{k})|\partial_{k_j}|u^\alpha(\mathbf{k})\rangle$ is symmetric in i,j

$$\langle u^\alpha(\mathbf{k})|\partial_{k_i}|u^\gamma(\mathbf{k})\rangle\langle u^\gamma(\mathbf{k})|\partial_{k_j}|u^\alpha(\mathbf{k})\rangle=\langle u^\alpha(\mathbf{k})|\partial_{k_j}|u^\gamma(\mathbf{k})\rangle\langle u^\gamma(\mathbf{k})|\partial_{k_i}|u^\alpha(\mathbf{k})\rangle$$

since α,γ is the summing indicator. besides $dk_i\wedge dk_j$ is anti-symmetric in i,j.

use the expression for the berry curvature, we can find that

$$\begin{aligned}\mathrm{tr}(\mathcal{F}^n)=&(\langle\partial_{k_{i_1}}u^\alpha(\mathbf{k})|\partial_{k_{j_1}}|u^{\eta_1}(\mathbf{k})\rangle)dk_{i_1}\wedge dk_{j_1}+\langle u^\alpha(\mathbf{k})|\partial_{k_{i_1}}|u^{\gamma_1}(\mathbf{k})\rangle\langle u^{\gamma_1}(\mathbf{k})|\partial_{k_{j_1}}|u^{\eta_1}(\mathbf{k})\rangle dk_{i_1}\wedge dk_{j_1}\\&\wedge(\langle\partial_{k_{i_2}}u^{\eta_1}(\mathbf{k})|\partial_{k_{j_2}}|u^{\eta_2}(\mathbf{k})\rangle)dk_{i_2}\wedge dk_{j_2}+\langle u^{\eta_1}(\mathbf{k})|\partial_{k_{i_2}}|u^{\gamma_2}(\mathbf{k})\rangle\langle u^{\gamma_2}(\mathbf{k})|\partial_{k_{j_2}}|u^{\eta_2}(\mathbf{k})\rangle dk_{i_2}\wedge dk_{j_2}\\&\wedge\cdots\\&\wedge(\langle\partial_{k_{i_2}}u^{\eta_{n-1}}(\mathbf{k})|\partial_{k_{j_n}}|u^\alpha(\mathbf{k})\rangle)dk_{i_n}\wedge dk_{j_n}+\langle u^{\eta_{n-1}}(\mathbf{k})|\partial_{k_{i_n}}|u^{\gamma_n}(\mathbf{k})\rangle\langle u^{\gamma_n}(\mathbf{k})|\partial_{k_{j_n}}|u^\alpha(\mathbf{k})\rangle dk_{i_n}\wedge dk_{j_n}\end{aligned}$$

if we write $F^{\alpha,\beta}_{i_n,j_n}=(\langle\partial_{k_{i_n}}u^\alpha(\mathbf{k})|\partial_{k_{j_n}}|u^\beta(\mathbf{k})\rangle+\langle u^\alpha(\mathbf{k})|\partial_{k_{i_n}}|u^\gamma(\mathbf{k})\rangle\langle u^\gamma(\mathbf{k})|\partial_{k_{j_n}}|u^\beta(\mathbf{k})\rangle)$, the above formula means that

$$\begin{aligned}\mathrm{tr}(\mathcal{F}^n)&=F^{\alpha,\beta_1}_{i_1,j_1}F^{\beta_1,\beta_2}_{i_2,j_2}\cdots F^{\beta_{n-1},\alpha}_{i_n,j_n}dk_{i_1}\wedge dk_{j_1}\wedge dk_{i_1}\wedge dk_{j_1}\wedge\cdots\wedge dk_{i_n}\wedge dk_{j_n}\\&=F^{\alpha,\beta_1}_{i_1,j_1}F^{\beta_1,\beta_2}_{i_2,j_2}\cdots F^{\beta_{n-1},\alpha}_{i_n,j_n}\epsilon_{i_1,j_1,i_2,j_2\cdots,i_n,j_n}dk_1\wedge dk_2\wedge dk_3\wedge dk_4\wedge\cdots\wedge dk_{2n-1}\wedge dk_{2n}\\&=(\sum_{\sigma\in S^{2n}}(-1)^\sigma F^{\alpha,\beta_1}_{\sigma(1),\sigma(2)}F^{\beta_1,\beta_2}_{\sigma(3),\sigma(4)}\cdots F^{\beta_{n-1},\alpha}_{\sigma(2n-1),\sigma(2n)})dk_1\wedge dk_2\wedge dk_3\wedge dk_4\wedge\cdots\wedge dk_{2n-1}\wedge dk_{2n}\\&=\hat{A}(F^{\alpha,\beta_1}_{1,2}F^{\beta_1,\beta_2}_{3,4}\cdots F^{\beta_{n-1},\alpha}_{2n-1,2n})dk_1\wedge dk_2\wedge dk_3\wedge dk_4\wedge\cdots\wedge dk_{2n-1}\wedge dk_{2n}\end{aligned}$$

where the $\hat{A}$ represent the anti-symmetric operator of 2n elements. in the flattened hamiltonian, we know that $Q=1-2P=1-2|u^\alpha(\mathbf{k})\rangle\langle u^\alpha(\mathbf{k})|$, so we have $dQ=-2dP=-2\partial_{k_i}(|u^\alpha(\mathbf{k})\rangle\langle u^\alpha(\mathbf{k})|)dk_i$, thus we have

$$\begin{aligned}Q(dQ)^{2n}&=(1-2P)(-2\partial_{k_{i_1}}P)(-2\partial_{k_{i_2}}P)\cdots(-2\partial_{k_{i_{2n}}}P)dk_{i_1}\wedge dk_{i_2}\wedge\cdots\wedge dk_{i_{2n}}\\&=(-2)^{2n}(1-2P)\partial_{k_{i_1}}P\partial_{k_{i_2}}P\cdots\partial_{k_{i_{2n}}}P\epsilon_{i_1,i_2,\cdots,i_{2n}}dk_1\wedge dk_2\wedge\cdots dk_{2n}\\&=(-2)^{2n}(1-2P)\hat{A}(\partial_1P\partial_2P\cdots\partial_{2n}P)dk_1\wedge dk_2\wedge\cdots dk_{2n}\end{aligned}$$

$$= ((-2)^{2n}\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P) + (-2)^{2n+1}P\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P))dk_1\wedge dk_2\wedge\cdots dk_{2n}$$

on the other hand, we have

$$\begin{aligned}
&\mathrm{tr}[\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P)]\\
&= \hat{A}\mathrm{tr}[\partial_1 P\partial_2 P\cdots\partial_{2n}P]\\
&= \hat{A}\mathrm{tr}[\partial_2 P\cdots\partial_{2n}P\partial_1 P]\\
&= \hat{A}\mathrm{tr}[\partial_{\sigma(1)}P\cdots\partial_{\sigma(2n-1)}P\partial_{\sigma(2n)}P] \qquad \sigma=(1,2,3,\cdots,2n)\in S^{2n}\\
&= (-1)^{\sigma}\mathrm{tr}[\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P)]\\
&= (-1)^{2n-1}\mathrm{tr}[\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P)]\\
&= -\mathrm{tr}[\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P)]\\
&\to \mathrm{tr}[\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P)] = 0
\end{aligned}$$

since then, we have

$$\mathrm{tr}(Q(dQ)^{2n}) = (-2)^{2n+1}\mathrm{tr}[P\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P)]dk_1\wedge dk_2\wedge\cdots dk_{2n}$$

so in order to prove

$$\mathrm{Ch}_{n=\frac{D}{2}} = \frac{1}{n!}(\frac{i}{2\pi})^n\int\mathrm{tr}(\mathcal{F}^n) = \frac{-1}{2^{2n+1}}\frac{1}{n!}(\frac{i}{2\pi})^n\int\mathrm{tr}(Q(dQ)^{2n})$$

we only need to prove that

$$\hat{A}(F_{1,2}^{\alpha,\beta_1}F_{3,4}^{\beta_1,\beta_2}\cdots F_{2n-1,2n}^{\beta_{n-1},\alpha}) = \mathrm{tr}[P\hat{A}(\partial_1 P\partial_2 P\cdots\partial_{2n}P)] = \hat{A}\mathrm{tr}[(P\partial_1 P\partial_2 P\cdots\partial_{2n}P)]$$

in the following, we prove the above formula

$$\begin{aligned}
&\mathrm{tr}[(P\partial_1 P\partial_2 P\cdots\partial_{2n}P)] = \langle u^\alpha|\partial_1(|u^{\beta_1}\rangle\langle u^{\beta_1}|)\partial_2(|u^{\beta_2}\rangle\langle u^{\beta_2}|)\cdots\partial_{2n}(|u^{\beta_{2n}}\rangle\langle u^{\beta_{2n}}|)|u^\alpha\rangle\\
&= \langle u^\alpha|(|\partial_1 u^{\beta_1}\rangle\langle u^{\beta_1}| + |u^{\beta_1}\rangle\langle\partial_1 u^{\beta_1}|)(|\partial_2 u^{\beta_2}\rangle\langle u^{\beta_2}| + |u^{\beta_2}\rangle\langle\partial_2 u^{\beta_2}|)\cdots(|\partial_{2n}u^{\beta_{2n}}\rangle\langle u^{\beta_{2n}}| + |u^{\beta_{2n}}\rangle\langle\partial_{2n}u^{\beta_{2n}}|)|u^\alpha\rangle\\
&= (\langle u^\alpha|\partial_1|u^{\beta_1}\rangle\langle u^{\beta_1}| + \delta^{\alpha,\beta_1}\langle\partial_1 u^{\beta_1}|)(|\partial_2 u^{\beta_2}\rangle\langle u^{\beta_2}| + |u^{\beta_2}\rangle\langle\partial_2 u^{\beta_2}|)\cdots(|\partial_{2n}u^{\beta_{2n}}\rangle\langle u^{\beta_{2n}}| + |u^{\beta_{2n}}\rangle\langle\partial_{2n}u^{\beta_{2n}}|)|u^\alpha\rangle\\
&= (\langle u^\alpha|\partial_1|u^{\beta_1}\rangle\langle u^{\beta_1}||\partial_2 u^{\beta_2}\rangle\langle u^{\beta_2}| + \langle u^\alpha|\partial_1|u^{\beta_1}\rangle\delta^{\beta_1,\beta_2}\langle\partial_2 u^{\beta_2}| + \langle\partial_1 u^\alpha||\partial_2 u^{\beta_2}\rangle\langle u^{\beta_2}| + \langle\partial_1 u^\alpha||u^{\beta_2}\rangle\langle\partial_2 u^{\beta_2}|)\partial_3 P\cdots\partial_{2n}P|u^\alpha\rangle\\
&= (\langle u^\alpha|\partial_1|u^{\beta_1}\rangle\langle u^{\beta_1}||\partial_2 u^{\beta_2}\rangle + \langle\partial_1 u^\alpha||\partial_2 u^{\beta_2}\rangle)\langle u^{\beta_2}|\partial_3 P\cdots\partial_{2n}P|u^\alpha\rangle\\
&= F_{1,2}^{\alpha,\beta_2}\langle u^{\beta_2}|\partial_3 P\cdots\partial_{2n}P|u^\alpha\rangle\\
&= F_{1,2}^{\alpha,\beta_2}F_{3,4}^{\beta_2,\beta_4}\langle u^{\beta_4}|\partial_5 P\cdots\partial_{2n}P|u^\alpha\rangle\\
&= F_{1,2}^{\alpha,\beta_2}F_{3,4}^{\beta_2,\beta_4}\cdots F_{2n-1,2n}^{\beta_{2n-2},\beta_{2n}}\langle u^{\beta_4}||u^\alpha\rangle\\
&= F_{1,2}^{\alpha,\beta_2}F_{3,4}^{\beta_2,\beta_4}\cdots F_{2n-1,2n}^{\beta_{2n-2},\beta_{2n}}\delta^{2n,\alpha}\\
&= F_{1,2}^{\alpha,\beta_2}F_{3,4}^{\beta_2,\beta_4}\cdots F_{2n-1,2n}^{\beta_{2n-2},\alpha}\\
&= F_{1,2}^{\alpha,\beta_1}F_{3,4}^{\beta_1,\beta_2}\cdots F_{2n-1,2n}^{\beta_{n-1},\alpha}
\end{aligned}$$

since the β_i is summing indicator represent the occupied bands, and in the above, we have used that

$$\langle u^\alpha|\partial_1|u^{\beta_1}\rangle\delta^{\beta_1,\beta_2}\langle\partial_2 u^{\beta_2}| + \langle\partial_1 u^\alpha||u^{\beta_2}\rangle\langle\partial_2 u^{\beta_2}| = \partial_1(\langle u^\alpha|u^{\beta_2}\rangle)\langle\partial_2 u^{\beta_2}| = 0$$

thus we have proved the equivalence.

so in the dimension D, we can write the Chern number as

$$\mathrm{Ch}_D = \frac{-1}{2^{D+1}}\frac{1}{\frac{D}{2}!}(\frac{i}{2\pi})^{\frac{D}{2}}\int\mathrm{tr}[Q(dQ)^D]$$

$$
\begin{aligned}
&= \frac{1}{V_D} \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})} \frac{-1}{2^{D+1}} \frac{1}{\frac{D}{2}!} (\frac{i}{2\pi})^{\frac{D}{2}} \int \mathrm{tr}[Q(dQ)^D] \\
&= -\frac{1}{V_D D!} (\frac{i}{2})^{\frac{D}{2}} \int \mathrm{tr}[Q(dQ)^D] \\
&= (-1)^{\frac{D-2}{2}} \frac{1}{V_D D!} (-\frac{i}{2})^{\frac{D}{2}} \int \mathrm{tr}[Q(dQ)^D]
\end{aligned}
$$

in the Dirac model, we know that

$$
\begin{aligned}
\mathrm{tr}[Q(dQ)^D &= \mathrm{tr}(\Gamma_{i_0}\Gamma_{i_1}\cdots\Gamma_{i_D}) n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D} \\
&= \mathrm{tr}(\Gamma_0\Gamma_1\cdots\Gamma_D)\epsilon_{i_0 i_1 \cdots i_D} n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D} \\
&= i^{\frac{D}{2}} \mathrm{tr}(\mathcal{I})\epsilon_{i_0 i_1 \cdots i_D} n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D}
\end{aligned}
$$

thus the Chern number is just

$$
\begin{aligned}
\mathrm{Ch}_D &= (-1)^{\frac{D-2}{2}} (-\frac{i}{2})^{\frac{D}{2}} i^{\frac{D}{2}} \mathrm{tr}(\mathcal{I}) \deg[\mathbf{n}] \\
&= (-1)^{\frac{D-2}{2}} (\frac{1}{2})^{\frac{D}{2}} \mathrm{tr}(\mathcal{I}) \deg[\mathbf{n}]
\end{aligned}
$$

for the primary series and the complex class, we know $n = [\frac{D+1}{2}] = \frac{D}{2}$, so $\mathrm{tr}(\mathcal{I}) = 2^{\frac{D}{2}}$, thus we have the expression for the Chern number

$$
\mathrm{Ch}_D = (-1)^{\frac{D-2}{2}} \deg[\mathbf{n}]
$$

for the even series, we know $n = [\frac{D+1}{2}] = \frac{D}{2} + 1$, thus we have

$$
\mathrm{Ch}_D = (-1)^{\frac{D-2}{2}} 2\deg[\mathbf{n}]
$$

these results are also slightly different from that one in the paper [1] by a factor $(-1)^{\frac{D-2}{2}}$, which I argue that it can be removed since it's a global factor equals to ± 1, which will not affect the classification.

$\Re$.5 The First Descendant Z_2 invariants represented as the wrapping number

for the first descendant, since they belongs to the real classes, we have followed the strategy by imposing the parity on $\mathbf{n}$ by

$$
\mathbf{n}(\mathbf{k}) = P\mathbf{n}(-\mathbf{k}) \qquad P = \mathrm{diag}\{+1, -1, -1, -1 \cdots, -1\}
$$

for the first Descendant for class X in D dimension, it can be derived as primary series for class X in D+1 dimension by imposing $n_{D+1}(\mathbf{k}) = 0$, although this can not be straightforward derived since they have the same symmetry, the only difference is the dimension of the base manifold $\mathbf{k}$.

we can think of it by considering $(k_1, k_2, \cdot, k_D) \equiv (k_1, k_2, \cdot, k_D, k_{D+1} = 0)$ in the D+1 dimensional primary series in the same symmetry class. since under image $n(k_1, k_2, \cdot, k_D)$ in the D+1 dimensional is D dimensional sphere in D+1 dimension, which can always be continuously deformed to the case where $n_{D+1}(\mathbf{k}) = 0$ for the hamiltonian $\sum_{i=0}^{D+1} n^i \Gamma_i$ parameterized by $n(k_1, k_2, \cdot, k_D, k_{D+1})$.

we consider two hamiltonians in the first descendants,$\mathbf{n}_1(\mathbf{k}), \mathbf{n}_2(\mathbf{k})$, then the path which connects this two hamiltonian in the D+1 dimension $\mathbf{n_{1\to 2}}(\mathbf{k}, t)$

$$
\mathbf{n_{1\to 2}}(\mathbf{k}, 0) = \mathbf{n}_1(\mathbf{k}), \mathbf{n_{1\to 2}}(\mathbf{k}, \pi) = \mathbf{n}_2(\mathbf{k})
$$

can be viewed as an elements in the primary series class hamiltonian, if we regard the extra time factor t as extra dimension k_{D+1} and imposing the required parity constrain

$$\mathbf{n_{1\to 2}}(\mathbf{k},t) = P\mathbf{n_{1\to 2}}(-\mathbf{k},-t) \qquad -\pi < t \leq 0$$

then we can define

$$\deg_2[\mathbf{n_{1\to 2}}] \equiv \deg[\mathbf{n_{1\to 2}}] \mod 2$$

thus we declare two hamiltonians $\mathbf{n}_1(\mathbf{k}), \mathbf{n}_2(\mathbf{k})$ are topologically equal if $\deg_2[\mathbf{n_{1\to 2}}] = 0$ and distinct if $\deg_2[\mathbf{n_{1\to 2}}] = 1$.

there is one thing for us to clarify, $\deg_2[\mathbf{n_{1\to 2}}]$ is independent of the chosen path. since the sign of Jacobian is $\text{sign}J_n \equiv 1 \mod 2$, and we can chose a reference point in S^{D+1}, namely the south pole $\mathbf{n}_0 = (-1,0,0,\cdots,0)$, if $(\mathbf{k},t)$ maps to the south pole, so does $(-\mathbf{k},-t)$, since $\mathbf{n_{1\to 2}}(\mathbf{k},t) = P\mathbf{n_{1\to 2}}(-\mathbf{k},-t)$, so if $(\mathbf{k},t) \neq (-\mathbf{k},-t)$, it will give us two points which is zero in the sense of mod 2, thus we only need to consider the high symmetry points in the BZ, we donated it as $(\bar{\mathbf{k}},0)$ and $(\bar{\mathbf{k}},\pi)$, where $\bar{\mathbf{k}}$ is the high symmetric point of the D dimensional BZ.

thus, we have

$$\begin{aligned}\deg_2[\mathbf{n_{1\to 2}}] &= \sum_{(\mathbf{k},t),\mathbf{n_{1\to 2}}(\mathbf{k},t)=\mathbf{n}_0} 1 \mod 2 \\ &= \sum_{\bar{\mathbf{k}},\bar{t}} \frac{1-n^0_{1\to 2}(\bar{\mathbf{k}},\bar{t})}{2} \mod 2 \\ &= \sum_{\bar{\mathbf{k}}} \frac{1-n^0_1(\bar{\mathbf{k}})}{2} + \frac{1-n^0_2(\bar{\mathbf{k}})}{2} \mod 2\end{aligned}$$

which only depends on $\mathbf{n}_1(\mathbf{k}), \mathbf{n}_2(\mathbf{k})$.

it's more convenient to use the parity instead of 0 and 1, that is

$$P_1[\mathbf{n}_1, \mathbf{n}_2] = (-1)^{\deg_2[\mathbf{n_{1\to 2}}]}$$

if we choose a reference map $\mathbf{n}_1(\mathbf{k}) = (1,0,0\cdots,0)$, then we define the parity of map $\mathbf{n}$ with respect to this parity:

$$P_1[\mathbf{n}] = P_1[\mathbf{n}_{ref}, \mathbf{n}] = (-1)^{\sum_{\bar{\mathbf{k}}} \frac{1-n^0(\bar{\mathbf{k}})}{2}} = \prod_{\bar{\mathbf{k}}} n^0(\bar{\mathbf{k}})$$

so for the map $\mathbf{n}(\mathbf{k})$, we can define

$$\deg_2[\mathbf{n}] = \sum_{\bar{\mathbf{k}}} \frac{1-n^0(\bar{\mathbf{k}})}{2}$$

where $\mathbf{n}$ is the map $T^D \to S^D$ by setting $n^{D+1} = 0$,and then we have

$$\deg_2[\mathbf{n_{1\to 2}}] = \deg[\mathbf{n}_1] + \deg[\mathbf{n}_2]$$

thus $\mathbf{n}_1(\mathbf{k}), \mathbf{n}_2(\mathbf{k})$ are topological equivalent if

$$\deg_2[\mathbf{n}_1] = \deg_2[\mathbf{n}_2]$$

and topological distinct if

$$\deg_2[\mathbf{n}_1] \neq \deg_2[\mathbf{n}_2]$$

and we can find that the parity of $\mathbf{n}$ is just

$$P_1[\mathbf{n}] = (-1)^{\deg_2[\mathbf{n}]}$$

ℜ.6 The Second Descendant Z_2 invariants represented as the wrapping number

for the Second Descendant in symmetry class X in D dimension, we can consider it as reduced from the the same class X from the primary series in D+2 dimension by imposing $n^{D+2} = n^{D+1} = 0$, consider the hamiltonian defined in dimension D,namely, $\mathbf{n}_1(\mathbf{k}), \mathbf{n}_2(\mathbf{k})$. then following the above strategy, we can define the path connecting this two hamiltonian $\mathbf{n}_{1\to2}(\mathbf{k}, s)$

$$\mathbf{n}_{1\to2}(\mathbf{k}, 0) = \mathbf{n}_1(\mathbf{k}) \qquad \mathbf{n}_{1\to2}(\mathbf{k}, \pi) = \mathbf{n}_2(\mathbf{k})$$

in which $\mathbf{n}_{1\to2}(\mathbf{k}, s) = P\mathbf{n}_{1\to2}(-\mathbf{k}, -s)$, this path define a map: $T^{D+1} \to S^{D+2}$, which is an elements in the first descendants in the symmetry class X. thus we can define

$$P_2[\mathbf{n}_1, \mathbf{n}_2] \equiv P_1[\mathbf{n}_{1\to2}]$$

which constitutes now a relative invariant between the two second descendant Hamiltonians. with the same argument above, this quantity is independent of path chosen, which only rely on $\mathbf{n}_1(\mathbf{k}), \mathbf{n}_2(\mathbf{k})$.

after chosen a reference hamiltonian $\mathbf{n}_{ref} = (1, 0, 0, \cdots)$, we can define the parity of the map $\mathbf{n}$

$$P_2[\mathbf{n}] = P_2[\mathbf{n}_{ref}, \mathbf{n}] = \prod_{\bar{\mathbf{k}}} n^0(\bar{\mathbf{k}}) = (-1)^{\deg_2[\mathbf{n}]}$$

where $\mathbf{n}$ is the map $T^D \to S^D$ by setting $n^{D+2} = n^{D+1} = 0$

§4 Real Space Universal topological Marker[2]

ℜ.1 The real space representation of the wrapping number

in the above, we have derived the Unified topological invariants, the wrapping number, namely

$$\begin{aligned}
\deg[\mathbf{n}] &= \frac{1}{V_D}\int_{\mathrm{BZ}} \frac{1}{D!}\epsilon_{i_0\cdots i_D} n^{i_0} dn^{i_1} \wedge dn^{i_2} \wedge \cdots \wedge dn^{i_D} \\
&= \frac{1}{V_D}\int_{\mathrm{BZ}} \epsilon_{i_0\cdots i_D} n^{i_0} \partial_1 n^{i_1} \partial_2 n^{i_2} \cdots \partial_D n^{i_D} d^D\mathbf{k} \\
&= \frac{1}{V_D}\int_{\mathrm{BZ}} \epsilon_{i_0\cdots i_D} r^{i_0} \frac{1}{|\mathbf{r}|^{D+1}} \partial_1 r^{i_1} \partial_2 r^{i_2} \cdots \partial_D r^{i_D} d^D\mathbf{k}
\end{aligned}$$

suppose the Dirac metrics used in the hamiltonian is $\{\Gamma_0, \Gamma_1, \cdots, \Gamma_D\}$ and left $\{\Gamma_{D+1}, \Gamma_{D+2}, \cdots, \Gamma_{2n}\}$ unused , we can define

$$W = \Gamma_{D+1}, \Gamma_{D+2}, \cdots, \Gamma_{2n}$$

then we can find that

$$\begin{aligned}
\mathrm{tr}[WQ(dQ)^D] &= \mathrm{tr}[\Gamma_{D+1}, \Gamma_{D+2}, \cdots, \Gamma_{2n}\Gamma_{i_0}, \Gamma_{i_1}, \cdots, \Gamma_{i_D}] n^{i_0} \partial_{j_1} n^{i_1} \partial_{j_2} n^{i_2} \cdots \partial_{j_D} n^{i_D} dk_{j_1} \wedge dk_{j_2} \wedge \cdots \wedge dk_{j_D} \\
&= \epsilon_{i_0, i_1, \cdots, i_D} 2^n i^n n^{i_0} \partial_{j_1} n^{i_1} \partial_{j_2} n^{i_2} \cdots \partial_{j_D} n^{i_D} \epsilon_{j_1, j_2, \cdots, j_D} dk_1 \wedge dk_2 \wedge \cdots \wedge dk_D \\
&= 2^n i^n D! \epsilon_{i_0, i_1, \cdots, i_D} n^{i_0} \partial_1 n^{i_1} \partial_2 n^{i_2} \cdots \partial_D n^{i_D} d^D\mathbf{k}
\end{aligned}$$

so we have

$$\begin{aligned}
&\epsilon_{i_0, i_1, \cdots, i_D} n^{i_0} \partial_1 n^{i_1} \partial_2 n^{i_2} \cdots \partial_D n^{i_D} d^D\mathbf{k} \\
&= \frac{1}{2^n i^n D!} \mathrm{tr}[WQ(dQ)^D] \\
&= \frac{1}{2^n i^n} \mathrm{tr}[WQ\partial_1 Q \partial_2 Q \cdots \partial_D Q] d^D\mathbf{k}
\end{aligned}$$

since then we can write the wrapping number as

$$\deg[\mathbf{n}] = \frac{(2\pi)^D}{V_D 2^n i^n} \int \frac{d^D\mathbf{k}}{(2\pi)^D} \mathrm{tr}[WQ\partial_1 Q\partial_2 Q \cdots \partial_D Q] \tag{3}$$

on the other hand, we can write the hamiltonian Q in terms of the projection of the valence band $p = \sum_{n,E_n<E_F} |n\rangle\langle n|$ and conduction band $q = p = \sum_{m,E_m>E_F} |m\rangle\langle m|$

$$Q = q - p \qquad q + p = I$$

in this way we can write the above formula to the real space one using the strategy that

$$\partial_j \to -iX_j \qquad \frac{d^D k}{(2\pi)^D} \to \frac{1}{L^D}$$

in the following, we work out the explicit form of the real space formula, in the odd D case, we have

$$\begin{aligned}
&WQ\partial_1 Q\partial_2 Q \cdots \partial_D Q \\
=&W(q-p)\partial_1 Q\partial_2 Q \cdots \partial_D Q \\
=&Wq\partial_1 Q\partial_2 Q \cdots \partial_D Q - Wp\partial_1 Q\partial_2 Q \cdots \partial_D Q \\
=&2^D(-1)^{\frac{D+1}{2}} Wq\partial_1 p\partial_2 q \cdots \partial_D p - 2^D(-1)^{\frac{D-1}{2}} Wp\partial_1 q\partial_2 p \cdots \partial_D q \\
=&(-1)^{\frac{D+1}{2}} 2^D W(q\partial_1 p\partial_2 q \cdots \partial_D p + Wp\partial_1 q\partial_2 p \cdots \partial_D q) \\
=&(-1)^{\frac{D+1}{2}} 2^D \sum_{m_1 \sim m_{\frac{D+1}{2}}} \sum_{n_1 \sim n_{\frac{D+1}{2}}} W\{(|m_1\rangle\langle m_1|)\partial_1(|n_1\rangle\langle n_1|)\partial_2(|m_2\rangle\langle m_2|)\partial_3(|n_2\rangle\langle n_2| \cdots \partial_D(|n_{\frac{D+1}{2}}\rangle\langle n_{\frac{D+1}{2}}|)) + (m \leftrightarrow n)\} \\
=&(-1)^{\frac{D+1}{2}} 2^D \sum_{m_1 \sim m_{\frac{D+1}{2}}} \sum_{n_1 \sim n_{\frac{D+1}{2}}} W\{|m_1\rangle\langle m_1|\partial_1|n_1\rangle\langle n_1|\partial_2|m_2\rangle\langle m_2|\partial_3|n_2\rangle\langle n_2| \cdots \partial_D|n_{\frac{D+1}{2}}\rangle\langle n_{\frac{D+1}{2}}| + (m \leftrightarrow n)\}
\end{aligned}$$

since $\partial_j Q = 2\partial_j q = -2\partial_j p$, where we use choose the form of $\partial_j Q$ accordingly so as to make sure the occurrence of p and q alternative.

then use the following identity

$$\langle m|\partial_j|n\rangle = -i\langle \Psi_m|X_j|\Psi_n\rangle \tag{4}$$

where

$$\langle r|\Psi_m(\mathbf{k})\rangle = \Psi_{m,\mathbf{k}}(r) = u_{m,\mathbf{k}}(r)e^{i\mathbf{k}\cdot r} = \langle r|m(\mathbf{k})\rangle e^{i\mathbf{k}\cdot r}$$

is the full wave function.

in order to prove this formula, we need to find the real space representation of the operator ∂_p, since we know $\hat{p} = -i\partial_x$ in the real space representation, namely $\langle x|\hat{p}|\psi\rangle = -i\partial_x\psi(x)$ from the fact that $\hat{p}$ is the generator of the translation operator in real space

$$\hat{T}_a|r\rangle = |r+a\rangle \qquad \hat{T}_\epsilon = I - i\epsilon\hat{p}$$

since

$$\begin{aligned}
\langle x|\hat{T}_\epsilon|\psi\rangle &= \langle \hat{T}^\dagger x||\psi\rangle = \langle x-\epsilon|\psi\rangle = \psi(x-\epsilon) \\
&= e^{-i(-i\epsilon\partial_x)}\psi(x) = (1 - i\epsilon(-i\partial_x))\langle x|\psi\rangle \equiv \langle x|(1-i\epsilon\hat{p})|\psi\rangle \\
&\to -i\partial_x\psi(x) = \langle x|\hat{p}|\psi\rangle
\end{aligned}$$

using the same strategy, we can find that $\hat{x} = i\partial_p$ in the momentum space representation of the operator $\hat{x}$ in the sense

$$i\partial_p \psi(p) = \langle p|\hat{x}|\psi\rangle$$

thus we can prove the formula (4).

$$\langle m|\partial_j|n\rangle = -i\langle m|X_j|n\rangle = -i\langle \Psi_m|X_j|\Psi_n\rangle$$

the second equality is due to the fact that X_j is diagonal in the real space representation.so that the extra phases factor can be cancelled since we evaluate it in the same point $\mathbf{k}$.

since then, we have:

$$\begin{aligned}
&WQ\partial_1 Q\partial_2 Q\cdots\partial_D Q\\
=&(-1)^{\frac{D+1}{2}}2^D \sum_{m_1\sim m_{\frac{D+1}{2}}}\sum_{n_1\sim n_{\frac{D+1}{2}}} W\{|m_1\rangle\langle m_1|\partial_1|n_1\rangle\langle n_1|\partial_2|m_2\rangle\langle m_2|\partial_3|n_2\rangle\langle n_2|\cdots\partial_D|n_{\frac{D+1}{2}}\rangle\langle n_{\frac{D+1}{2}}| + (m\leftrightarrow n)\}\\
=&(-1)^{\frac{D+1}{2}}2^D \sum_{m_1\sim m_{\frac{D+1}{2}}}\sum_{n_1\sim n_{\frac{D+1}{2}}} W\{|\Psi_{m_1}\rangle\langle\Psi_{m_1}| - iX_1|\Psi_{n_1}\rangle\langle\Psi_{n_1}| - iX_2|\Psi_{m_2}\rangle\langle\Psi_{m_2}| - iX_3|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots\\
&\cdots - iX_D|\Psi_{n_{\frac{D+1}{2}}}\rangle\langle\Psi_{n_{\frac{D+1}{2}}}| + (m\leftrightarrow n)\}\\
=&i2^D \sum_{m_1\sim m_{\frac{D+1}{2}}}\sum_{n_1\sim n_{\frac{D+1}{2}}} W\{|\Psi_{m_1}\rangle\langle\Psi_{m_1}|X_1|\Psi_{n_1}\rangle\langle\Psi_{n_1}|X_2|\Psi_{m_2}\rangle\langle\Psi_{m_2}|X_3|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots\\
&\cdots X_D|\Psi_{n_{\frac{D+1}{2}}}\rangle\langle\Psi_{n_{\frac{D+1}{2}}}| + (m\leftrightarrow n)\}
\end{aligned}$$

thus we have

$$\begin{aligned}
&\int\frac{d^D\mathbf{k}}{(2\pi)^D}\mathrm{tr}[WQ\partial_1 Q\partial_2 Q\cdots\partial_D Q]\\
&= i2^D\int\frac{d^D\mathbf{k}}{(2\pi)^D}\mathrm{tr}[\sum_{m_1\sim m_{\frac{D+1}{2}}}\sum_{n_1\sim n_{\frac{D+1}{2}}} W\{|\Psi_{m_1}\rangle\langle\Psi_{m_1}|X_1|\Psi_{n_1}\rangle\langle\Psi_{n_1}|X_2|\Psi_{m_2}\rangle\langle\Psi_{m_2}|X_3|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots\\
&\cdots X_D|\Psi_{n_{\frac{D+1}{2}}}\rangle\langle\Psi_{n_{\frac{D+1}{2}}}| + (m\leftrightarrow n)\}]\\
&= i2^D\int\frac{d^D\mathbf{k}}{(2\pi)^D}\mathrm{tr}[W\{\sum_{m_1}|\Psi_{m_1}\rangle\langle\Psi_{m_1}|X_1\sum_{n_1}|\Psi_{n_1}\rangle\langle\Psi_{n_1}|X_2\sum_{m_2}|\Psi_{m_2}\rangle\langle\Psi_{m_2}|X_3\sum_{n_2}|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots\\
&\cdots X_D\sum_{n_{\frac{D+1}{2}}}|\Psi_{n_{\frac{D+1}{2}}}\rangle\langle\Psi_{n_{\frac{D+1}{2}}}| + (m\leftrightarrow n)\}]\\
&= i2^D\mathrm{tr}[W\{\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{m_1}|\Psi_{m_1}\rangle\langle\Psi_{m_1}|X_1\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{n_1}|\Psi_{n_1}\rangle\langle\Psi_{n_1}|X_2\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{m_2}|\Psi_{m_2}\rangle\langle\Psi_{m_2}|X_3\\
&\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{n_2}|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots X_D\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{n_{\frac{D+1}{2}}}|\Psi_{n_{\frac{D+1}{2}}}\rangle\langle\Psi_{n_{\frac{D+1}{2}}}| + (m\leftrightarrow n)\}]\\
&= \frac{1}{L^D}i2^D\mathrm{tr}[WQX_1PX_2Q\cdots QX_DP + WPX_1QX_2P\cdots PX_DQ]
\end{aligned}$$

where the factor $\frac{1}{L^D}$ comes from the correspondence $\int\frac{d^D\mathbf{k}}{(2\pi)^D}\to\frac{1}{L^D}$ which one is the average over the first BZ in momentum space and the other one is average over the unite cell in real space. besides,

$$Q = \int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_m|\Psi_m\rangle\langle\Psi_m| = \sum_m|E_m\rangle\langle E_m| \quad P = \int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_n|\Psi_n\rangle\langle\Psi_n| = \sum_n|E_n\rangle\langle E_n|$$

to represent the projection to the unoccupied (occupied) energy state of the lattice hamiltonian since they are diagonal in the band level. and sum over k is just the whole tr of the big lattice hamiltonian.

in the above, we have used the fact that[4]

$$\langle\Psi_{n,k}|\hat{x}|\Psi_{n',k'}\rangle = i\delta_{n,n'}\delta_{k,k'}\frac{\partial}{\partial_k} + \delta_{k,k'}iN\int e^{i(k'-k)x}u^*_{n,k}(x)\frac{\partial}{\partial_k}u_{n',k'}(x)dx$$

so that if $k \neq k'$, the extra matrix elements we have added is equal to zero.

the above formula can be derived from the following process

$$\begin{aligned}
\langle\Psi_{n,k}|\hat{x}|\Psi_{n',k'}\rangle &= \int dx\langle\Psi_{n,k}|\hat{x}|x\rangle\langle x||\Psi_{n',k'}\rangle \\
&= \int dxxu^*_{n,k}(x)e^{-ikx}u_{n',k'}(x)e^{ik'x} \\
&= \int dxi\frac{\partial}{\partial k}(e^{-ix(k-k')})u^*_{n,k}(x)u_{n',k'}(x) \\
&= i\frac{\partial}{\partial k}(\int dxe^{-ix(k-k')}u^*_{n,k}(x)u_{n',k'}(x)) - i\int dxe^{-ix(k-k')}\frac{\partial}{\partial k}(u^*_{n,k}(x))u_{n',k'}(x) \\
&= i\frac{\partial}{\partial k}\delta_{n,n'}\delta_{k,k'} - i\int dxe^{ix(k'-k)}\frac{\partial}{\partial k}(u^*_{n,k}(x))u_{n',k'}(x)
\end{aligned}$$

and the second term is proportional to $\delta_{k,k'}$ since $\frac{\partial}{\partial k}(u^*_{n,k}(x))u_{n',k'}(x)$ can be think as independent of x due to it's periodic with periodicity lattice constant which is a small scale in the thermal dynamic limit.

for the case D is even, we have

$$\begin{aligned}
&WQ\partial_1Q\partial_2Q\cdots\partial_DQ \\
=&W(q-p)\partial_1Q\partial_2Q\cdots\partial_DQ \\
=&Wq\partial_1Q\partial_2Q\cdots\partial_DQ - Wp\partial_1Q\partial_2Q\cdots\partial_DQ \\
=&2^D(-1)^{\frac{D}{2}}Wq\partial_1p\partial_2q\cdots\partial_Dq - 2^D(-1)^{\frac{D}{2}}Wp\partial_1q\partial_2p\cdots\partial_Dp \\
=&(-1)^{\frac{D}{2}}2^DW(q\partial_1p\partial_2q\cdots\partial_Dq - Wp\partial_1q\partial_2p\cdots\partial_Dp) \\
=&(-1)^{\frac{D}{2}}2^D\sum_{m_1\sim m_{\frac{D}{2}+1}}\sum_{n_1\sim n_{\frac{D}{2}}}W\{(|m_1\rangle\langle m_1|)\partial_1(|n_1\rangle\langle n_1|)\partial_2(|m_2\rangle\langle m_2|)\partial_3(|n_2\rangle\langle n_2|\cdots\partial_D(|m_{\frac{D}{2}+1}\rangle\langle m_{\frac{D}{2}+1}|)) - (m\leftrightarrow n)\} \\
=&(-1)^{\frac{D}{2}}2^D\sum_{m_1\sim m_{\frac{D}{2}+1}}\sum_{n_1\sim n_{\frac{D}{2}}}W\{|m_1\rangle\langle m_1|\partial_1|n_1\rangle\langle n_1|\partial_2|m_2\rangle\langle m_2|\partial_3|n_2\rangle\langle n_2|\cdots\partial_D|m_{\frac{D}{2}+1}\rangle\langle m_{\frac{D}{2}+1}| - (m\leftrightarrow n)\} \\
=&(-1)^{\frac{D}{2}}2^D\sum_{m_1\sim m_{\frac{D}{2}+1}}\sum_{n_1\sim n_{\frac{D}{2}}}W\{|\Psi_{m_1}\rangle\langle\Psi_{m_1}| - iX_1|\Psi_{n_1}\rangle\langle\Psi_{n_1}| - iX_2|\Psi_{m_2}\rangle\langle\Psi_{m_2}| - iX_3|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots \\
&\cdots - iX_D|\Psi_{m_{\frac{D}{2}+1}}\rangle\langle\Psi_{m_{\frac{D}{2}+1}}| - (m\leftrightarrow n)\} \\
=&2^D\sum_{m_1\sim m_{\frac{D}{2}+1}}\sum_{n_1\sim n_{\frac{D}{2}}}W\{|\Psi_{m_1}\rangle\langle\Psi_{m_1}|X_1|\Psi_{n_1}\rangle\langle\Psi_{n_1}|X_2|\Psi_{m_2}\rangle\langle\Psi_{m_2}|X_3|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots \\
&\cdots X_D|\Psi_{m_{\frac{D}{2}+1}}\rangle\langle\Psi_{m_{\frac{D}{2}+1}}| - (m\leftrightarrow n)\}
\end{aligned}$$

thus we have

$$\begin{aligned}
&\int\frac{d^D\mathbf{k}}{(2\pi)^D}\mathrm{tr}[WQ\partial_1Q\partial_2Q\cdots\partial_DQ] \\
&= 2^D\int\frac{d^D\mathbf{k}}{(2\pi)^D}\mathrm{tr}[\sum_{m_1\sim m_{\frac{D}{2}+1}}\sum_{n_1\sim n_{\frac{D}{2}}}W\{|\Psi_{m_1}\rangle\langle\Psi_{m_1}|X_1|\Psi_{n_1}\rangle\langle\Psi_{n_1}|X_2|\Psi_{m_2}\rangle\langle\Psi_{m_2}|X_3|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots
\end{aligned}$$

$$\cdots X_D|\Psi_{m_{\frac{D}{2}+1}}\rangle\langle\Psi_{m_{\frac{D}{2}+1}}| - (m \leftrightarrow n)\}]$$

$$= 2^D \int \frac{d^D\mathbf{k}}{(2\pi)^D}\mathrm{tr}[W\{\sum_{m_1}|\Psi_{m_1}\rangle\langle\Psi_{m_1}|X_1\sum_{n_1}|\Psi_{n_1}\rangle\langle\Psi_{n_1}|X_2\sum_{m_2}|\Psi_{m_2}\rangle\langle\Psi_{m_2}|X_3\sum_{n_2}|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots$$

$$\cdots X_D\sum_{m_{\frac{D}{2}+1}}|\Psi_{m_{\frac{D}{2}+1}}\rangle\langle\Psi_{m_{\frac{D}{2}+1}}| - (m \leftrightarrow n)\}]$$

$$= 2^D\mathrm{tr}[W\{\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{m_1}|\Psi_{m_1}\rangle\langle\Psi_{m_1}|X_1\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{n_1}|\Psi_{n_1}\rangle\langle\Psi_{n_1}|X_2\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{m_2}|\Psi_{m_2}\rangle\langle\Psi_{m_2}|X_3$$

$$\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{n_2}|\Psi_{n_2}\rangle\langle\Psi_{n_2}|\cdots\cdots X_D\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{m_{\frac{D}{2}+1}}|\Psi_{m_{\frac{D}{2}+1}}\rangle\langle\Psi_{m_{\frac{D}{2}+1}}| - (m \leftrightarrow n)\}]$$

$$= \frac{1}{L^D}2^D\mathrm{tr}[WQX_1PX_2Q\cdots PX_DQ - WPX_1QX_2P\cdots QX_DP]$$

so the wrapping number can be written as

$$\deg[\mathbf{n}] = \frac{1}{L^D}\frac{(2\pi)^D}{V_D 2^n i^n}i2^D\mathrm{tr}[WQX_1PX_2Q\cdots QX_DP + WPX_1QX_2P\cdots PX_DQ] = \frac{1}{L^D}\mathrm{tr}[\hat{C}] \tag{5}$$

for the case D is odd, and can be written as

$$\deg[\mathbf{n}] = \frac{1}{L^D}\frac{(2\pi)^D}{V_D 2^n i^n}2^D\mathrm{tr}[WQX_1PX_2Q\cdots PX_DQ - WPX_1QX_2P\cdots QX_DP] = \frac{1}{L^D}\mathrm{tr}[\hat{C}] \tag{6}$$

for the case D is even, where $\hat{C}$ is named as the topological operator. using the topological operator, we can define the local and non-local topological marker as

$$C(r) \equiv \langle r|\hat{C}|r\rangle$$
$$C(r, r') \equiv \langle r|\hat{C}|r'\rangle$$

so in three dimension, the topological operator can be written as

$$\hat{C}_3 = i\frac{(2\pi)^D}{V_D}2^D\frac{1}{2^n i^n}W(QXPYQZP + PXQYPZQ) = i\frac{32\pi}{2^n i^n}W(QXPYQZP + PXQYPZQ)$$

where n and W depends on different symmetry class.

in two dimension, the topological operator can be written as

$$\hat{C}_2 = \frac{(2\pi)^D}{V_D}2^D\frac{1}{2^n i^n}W(QXPYQ - PXQYP) = \frac{4\pi}{2^n i^n}W(QXPYQ - PXQYP)$$

similarly, n and W depends on different symmetry class.

in one dimension, the topological operator can be written as

$$\hat{C}_1 = i\frac{(2\pi)^D}{V_D}2^D\frac{1}{2^n i^n}W(QXP + PXQ) = i\frac{2}{2^n i^n}W(QXP + PXQ)$$

in the following, we make some comments on the above results, we should notice that the factor $2^n i^n$ comes from the product of all the Dirac matrices

$$\mathrm{tr}[\Gamma_{D+1}, \Gamma_{D+2}, \cdots, \Gamma_{2n}\Gamma_{i_0}, \Gamma_{i_1}, \cdots, \Gamma_{i_D}]$$

so in practical calculation, we should organize the order of the Gamma matrices in W such that the above formula holds, or equivalently, replace the factor $2^n i^n$ with $\mathrm{tr}[\Gamma_{D+1}, \Gamma_{D+2}, \cdots, \Gamma_{2n}\Gamma_0, \Gamma_1, \cdots, \Gamma_D]$, since 2^n comes from the dimension of the Gamma matrices, so we should only replace i^n with the scalar factor of $\Gamma_{D+1}, \Gamma_{D+2}, \cdots, \Gamma_{2n}\Gamma_0, \Gamma_1, \cdots, \Gamma_D = W\Gamma_0, \Gamma_1, \cdots, \Gamma_D$.

on the other hand, in the above formula, we have used the formula

$$\langle m|\partial_j|n\rangle = -i\langle\Psi_m|X_j|\Psi_n\rangle \tag{7}$$

which is valid for the infinity and continuous real space, if we use the finite lattice model with the periodic boundary condition, we should use the exponential position operator, namely

$$\langle m|\partial_j|n\rangle = \frac{L}{2\pi}\langle m|e^{i\frac{2\pi}{L}X_j}|n\rangle \tag{8}$$

since we know

$$\begin{aligned}\langle m|\partial_j|n\rangle &= \langle m|\frac{|n_{k+\delta k}\rangle - |n_k\rangle}{\delta k} = \frac{1}{\delta k}\langle m|e^{\partial_k\cdot\delta k}|n\rangle \\ &= \frac{1}{\delta k}\langle m|e^{iX_j\cdot\delta k}|n\rangle = \frac{L}{2\pi}\langle m|e^{i\frac{2\pi}{L}X_j}|n\rangle \\ &= \langle m|\frac{L}{2\pi}e^{i\frac{2\pi}{L}X_j}|n\rangle\end{aligned}$$

$\frac{L}{2\pi}e^{i\frac{2\pi}{L}X_j}$ is called the exponential position operator, if we use this one, the wrapping number should modified as

$$\deg[\mathbf{n}] \to \frac{1}{(-i)^D}\deg[\mathbf{n}]$$

$$\deg[\mathbf{n}] = i^{D+1}\frac{1}{L^D}\frac{(2\pi)^D}{V_D 2^n i^n}2^D\mathrm{tr}[WQX_1PX_2Q\cdots QX_DP + WPX_1QX_2P\cdots PX_DQ] = \frac{1}{L^D}\mathrm{tr}[\hat{C}] \tag{9}$$

for the case D is odd, and can be written as

$$\deg[\mathbf{n}] = i^{D}\frac{1}{L^D}\frac{(2\pi)^D}{V_D 2^n i^n}2^D\mathrm{tr}[WQX_1PX_2Q\cdots PX_DQ - WPX_1QX_2P\cdots QX_DP] = \frac{1}{L^D}\mathrm{tr}[\hat{C}] \tag{10}$$

for the even D case.

ℜ.2 Explicit form for different symmetry classes in three dimension

∮.1 3D-AIII

for 3D AIII class, the Dirac metrics is $2^n = 2^{\frac{D+1}{2}} = 4$ dimension, then the five Dirac metrics are given by [5]

$$\gamma_1 = \begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix} \quad \gamma_2 = \begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix} \quad \gamma_3 = \begin{pmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{pmatrix}$$
$$\gamma_4 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \gamma_5 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

the chiral operator is chosen as $S = \gamma_4$, thus the hamiltonian is given by

$$H(k) = Ak_x\gamma_1 + Ak_y\gamma_2 + Ak_z\gamma_3 + (M + B\sum_{l=1}^{3}k_l^2)\gamma_5$$

The spinor of 3D class AIII contains only annihilation operators, which we name generically as $\Psi = (c_{\mathbf{k},1}, c_{\mathbf{k},2}, c_{\mathbf{k},3}, c_{\mathbf{k},4})^T$. in order to translate it to the lattice tight binding model, we have to make the replacement $k_l = \sin(k_l)$, $k_l^2 = 2(1-\cos(k_l))$, thus the hamiltonian can be written as(in the lattice model)

$$H(k) = Ak_x\gamma_1 + Ak_y\gamma_2 + Ak_z\gamma_3 + (M + B\sum_{l=1}^{3}k_l^2)\gamma_5$$

$$= A\sin k_x\gamma_1 + A\sin k_y\gamma_2 + A\sin k_z\gamma_3 + (M + B\sum_{l=1}^{3} 2(1-\cos k_l))\gamma_5$$
$$= A\sin k_x\gamma_1 + A\sin k_y\gamma_2 + A\sin k_z\gamma_3 + (M+6B)\gamma_5 - 2B\sum_{l=1}^{3}\cos k_l\gamma_5$$

in order to translate this momentum hamiltonian to the real space lattice hamiltonian, we have to do Fourier transform

$$c_{\mathbf{k},I} = \frac{1}{\sqrt{N}}\sum_i e^{-i\mathbf{r}_i\cdot\mathbf{k}}c_{i,I} \qquad c_{i,I} = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i\mathbf{r}_i\cdot\mathbf{k}}c_{\mathbf{k},I}$$

so we have

$$\begin{aligned}\sum_i c^\dagger_{i+s,I}c_{i,J} &= \frac{1}{N}\sum_{i,\mathbf{k},\mathbf{k}'} e^{-i\mathbf{r}_{i+s}\cdot\mathbf{k}}e^{i\mathbf{r}_i\cdot\mathbf{k}'}c^\dagger_{\mathbf{k},I}c_{\mathbf{k}',J}\\ &= \frac{1}{N}\sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{r}_s\cdot\mathbf{k}}\sum_i e^{-i\mathbf{r}_i\cdots(\mathbf{k}-\mathbf{k}')}c^\dagger_{\mathbf{k},I}c_{\mathbf{k}',J}\\ &= \frac{1}{N}\sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{r}_s\cdot\mathbf{k}}N\delta_{\mathbf{k},\mathbf{k}'}c^\dagger_{\mathbf{k},I}c_{\mathbf{k}',J}\\ &= \sum_{\mathbf{k}} e^{-i\mathbf{r}_s\cdot\mathbf{k}}c^\dagger_{\mathbf{k},I}c_{\mathbf{k}',J}\end{aligned}$$

using the above formula, we find

$$\sum_{\mathbf{k}}\sin k_l c^\dagger_{\mathbf{k},I}c_{\mathbf{k},J} = \frac{i}{2}\sum_i(c^\dagger_{i+X_l,I}c_{i,J} - c^\dagger_{i-X_l,I}c_{i,J}) = \frac{i}{2}\sum_i(c^\dagger_{i+X_l,I}c_{i,J} - c^\dagger_{i,I}c_{i+X_l,J})$$
$$\sum_{\mathbf{k}}\cos k_l c^\dagger_{\mathbf{k},I}c_{\mathbf{k},J} = \frac{1}{2}\sum_i(c^\dagger_{i+X_l,I}c_{i,J} + c^\dagger_{i-X_l,I}c_{i,J}) = \frac{1}{2}\sum_i(c^\dagger_{i+X_l,I}c_{i,J} + c^\dagger_{i,I}c_{i+X_l,J})$$

so we have

$$\begin{aligned}&A\sin k_x\gamma_1\\ &= A\sin k_x(c^\dagger_{\mathbf{k},1}c_{\mathbf{k},4} + c^\dagger_{\mathbf{k},2}c_{\mathbf{k},3} + c^\dagger_{\mathbf{k},3}c_{\mathbf{k},2} + c^\dagger_{\mathbf{k},4}c_{\mathbf{k},1})\\ &= A\frac{i}{2}\sum_i\sum_{(I,J)=(1,4),(2,3),(3,2),(4,1)}(c^\dagger_{i+x,I}c_{i,J} - c^\dagger_{i,I}c_{i+x,J})\\ &= A\frac{i}{2}\sum_i\{(c^\dagger_{i+x,1}c_{i,4} - c^\dagger_{i,1}c_{i+x,4}) + (c^\dagger_{i+x,2}c_{i,3} - c^\dagger_{i,2}c_{i+x,3}) + (c^\dagger_{i+x,3}c_{i,2} - c^\dagger_{i,3}c_{i+x,2}) + (c^\dagger_{i+x,4}c_{i,1} - c^\dagger_{i,4}c_{i+x,1})\}\end{aligned}$$

$$\begin{aligned}&A\sin k_y\gamma_2\\ &= A\sin k_y(-ic^\dagger_{\mathbf{k},1}c_{\mathbf{k},4} + ic^\dagger_{\mathbf{k},2}c_{\mathbf{k},3} - ic^\dagger_{\mathbf{k},3}c_{\mathbf{k},2} + ic^\dagger_{\mathbf{k},4}c_{\mathbf{k},1})\\ &= A\frac{i}{2}\sum_i\{-i(c^\dagger_{i+y,1}c_{i,4} - c^\dagger_{i,1}c_{i+y,4}) + i(c^\dagger_{i+y,2}c_{i,3} - c^\dagger_{i,2}c_{i+y,3}) - i(c^\dagger_{i+y,3}c_{i,2} - c^\dagger_{i,3}c_{i+y,2}) + i(c^\dagger_{i+y,4}c_{i,1} - c^\dagger_{i,4}c_{i+y,1})\}\end{aligned}$$

$$\begin{aligned}&A\sin k_z\gamma_3\\ &= A\sin k_z(c^\dagger_{\mathbf{k},1}c_{\mathbf{k},3} - c^\dagger_{\mathbf{k},2}c_{\mathbf{k},4} + c^\dagger_{\mathbf{k},3}c_{\mathbf{k},1} - c^\dagger_{\mathbf{k},4}c_{\mathbf{k},2})\\ &= A\frac{i}{2}\sum_i\{(c^\dagger_{i+z,1}c_{i,3} - c^\dagger_{i,1}c_{i+z,3}) - (c^\dagger_{i+z,2}c_{i,4} - c^\dagger_{i,2}c_{i+z,4}) + (c^\dagger_{i+z,3}c_{i,1} - c^\dagger_{i,3}c_{i+z,1}) - (c^\dagger_{i+z,4}c_{i,2} - c^\dagger_{i,4}c_{i+z,2})\}\end{aligned}$$

$$
\begin{aligned}
&(M+6B)\gamma_5 \\
&=(M+6B)(-ic_{\mathbf{k},1}^{\dagger}c_{\mathbf{k},3}-ic_{\mathbf{k},2}^{\dagger}c_{\mathbf{k},4}+ic_{\mathbf{k},3}^{\dagger}c_{\mathbf{k},1}+ic_{\mathbf{k},4}^{\dagger}c_{\mathbf{k},2}) \\
&=-i(M+6B)\sum_i\{c_{i,1}^{\dagger}c_{i,3}+c_{i,2}^{\dagger}c_{i,4}-c_{i,3}^{\dagger}c_{i,1}-c_{i,4}^{\dagger}c_{i,2}\}
\end{aligned}
$$

$$
\begin{aligned}
&-2B\sum_l\cos k_l\gamma_5 \\
&=-2B\sum_l\cos k_l(-ic_{\mathbf{k},1}^{\dagger}c_{\mathbf{k},3}-ic_{\mathbf{k},2}^{\dagger}c_{\mathbf{k},4}+ic_{\mathbf{k},3}^{\dagger}c_{\mathbf{k},1}+ic_{\mathbf{k},4}^{\dagger}c_{\mathbf{k},2}) \\
&=i2B\sum_l\frac{1}{2}\{(c_{i+X_l,1}^{\dagger}c_{i,3}+c_{i,1}^{\dagger}c_{i+X_l,3})+(c_{i+X_l,2}^{\dagger}c_{i,4}+c_{i,2}^{\dagger}c_{i+X_l,4})-(c_{i+X_l,3}^{\dagger}c_{i,1}+c_{i,3}^{\dagger}c_{i+X_l,1}) \\
&-(c_{i+X_l,4}^{\dagger}c_{i,2}+c_{i,4}^{\dagger}c_{i+X_l,2})\}
\end{aligned}
$$

collecting all the terms, we can write down the real space lattice hamiltonian as

$$
\begin{aligned}
&i\frac{A}{2}\sum_i\{c_{i+x,1}^{\dagger}c_{i,4}-c_{i,1}^{\dagger}c_{i+x,4}+c_{i+x,2}^{\dagger}c_{i,3}-c_{i,2}^{\dagger}c_{i+x,3}\} \\
&+i\frac{A}{2}\sum_i\{-i(c_{i+y,1}^{\dagger}c_{i,4}-c_{i,1}^{\dagger}c_{i+y,4})+i(c_{i+y,2}^{\dagger}c_{i,3}-c_{i,2}^{\dagger}c_{i+y,3})\} \\
&+i\frac{A}{2}\sum_i\{(c_{i+z,1}^{\dagger}c_{i,3}-c_{i,1}^{\dagger}c_{i+z,3})-(c_{i+z,2}^{\dagger}c_{i,4}-c_{i,2}^{\dagger}c_{i+z,4})\} \\
&-i(M+6B)\sum_i\{c_{i,1}^{\dagger}c_{i,3}+c_{i,2}^{\dagger}c_{i,4}\} \\
&\ i2B\frac{1}{2}\sum_i\sum_{X_l=x,y,z}\{(c_{i+X_l,1}^{\dagger}c_{i,3}+c_{i,1}^{\dagger}c_{i+X_l,3})+(c_{i+X_l,2}^{\dagger}c_{i,4}+c_{i,2}^{\dagger}c_{i+X_l,4})\} \\
&+H.c
\end{aligned}
$$

in this case, we have n=2, W is the chiral operator $W=\gamma_4$, and we have

$$2^n i^n\to\mathrm{tr}[\Gamma_4\Gamma_5\Gamma_1\Gamma_2\Gamma_3]=2^2$$

so the topological operator read as

$$\hat{C}_{3D-AIII}=i\frac{32\pi}{2^n i^n}W(QXPYQZP+PXQYPZQ)=8\pi i\gamma_4(QXPYQZP+PXQYPZQ)$$

∮.2 3D-DIII-The B phase of superfluid ^3He

A concrete example of 3D class DIII is the B phase of superfluid ^{3}He[6][7]. in this case, we use the representation of Dirac matrices in the Bernevig-Hughes-Zhang (BHZ) model[8][9]

$$\Gamma_{1\sim5}=\{s_x\otimes\sigma_z,s_y\otimes I,s_z\otimes I,s_x\otimes\sigma_x,s_x\otimes\sigma_y\}$$

where σ_i acts on the spin space and s_i acts on the particle-hole space. we can find that there are 3 real matrices and 2 purely imaginary matrices , which meet the requirement of the well-defined B

$$B=(s_x\otimes\sigma_z)(s_z\otimes I)(s_x\otimes\sigma_x)=-is_z\otimes\sigma_y$$

if we choose $\gamma_0=s_z\otimes I$, the $A=B\gamma_0=-I\otimes i\sigma_y$ which is the time reversal operator of the Dirac hamiltonian,namely $T=-I\otimes i\sigma_yK$, and if we choose the chiral operator to be $S=s_x\otimes\sigma_y$,then the particle-hole operator is

$$AS=(-I\otimes i\sigma_y)(s_x\otimes\sigma_y)=-is_x\otimes I\to C=-is_x\otimes IK$$

then the Dirac hamiltonian in the momentum space can be written as

$$
\begin{aligned}
H(k) &= \Delta \sin k_x s_x \otimes \sigma_z + \Delta \sin k_y s_y \otimes I + (-\Delta \sin k_z) s_x \otimes \sigma_x \\
&+ \{2t(\cos k_x + \cos k_y + \cos k_z) - \mu\} s_z \otimes I
\end{aligned}
$$

in order to convert this model to the real space lattice, we need to figure out the basis, which written in the Nambu spinor as

$$\eta_{\mathbf{k}}^\dagger = (c_{\mathbf{k},\uparrow}^\dagger, c_{-\mathbf{k},\uparrow}, c_{\mathbf{k},\downarrow}^\dagger, c_{-\mathbf{k},\downarrow})$$

where $c_{\mathbf{k},\uparrow}^\dagger$ is the electron-like creation operator with momentum $\mathbf{k}$ and spin $\uparrow$, $c_{-\mathbf{k},\uparrow}$ is the hole-like creation operator with momentum $-\mathbf{k}$ and spin $\uparrow$. if we consider it in the Fourier transform sense, we can find that

$$
\begin{aligned}
c_{\mathbf{k},\uparrow} &= \frac{1}{\sqrt{N}} \sum_j e^{-i\mathbf{R}_j \cdot \mathbf{k}} c_{j,\uparrow} \\
c_{\mathbf{k},\uparrow}^\dagger &= \frac{1}{\sqrt{N}} \sum_j e^{i\mathbf{R}_j \cdot \mathbf{k}} c_{j,\uparrow}^\dagger \\
c_{-\mathbf{k},\uparrow}^\dagger &= \frac{1}{\sqrt{N}} \sum_j e^{-i\mathbf{R}_j \cdot \mathbf{k}} c_{j,\uparrow}^\dagger
\end{aligned}
$$

where $c_{j,\uparrow}$ is the electron-like annihilation operator on lattice site $\mathbf{R}_j$ with spin $\uparrow$ and $c_{j,\uparrow}^\dagger$ is the hole-like annihilation operator on lattice site $\mathbf{R}_j$ with spin $\uparrow$

the two set of basis $(c_{\mathbf{k},\uparrow}, c_{-\mathbf{k},\uparrow}^\dagger, c_{\mathbf{k},\downarrow}, c_{-\mathbf{k},\downarrow}^\dagger)$, $(c_{j,\uparrow}, c_{j,\uparrow}^\dagger, c_{j,\downarrow}, c_{j,\downarrow}^\dagger)$ are connected by the usual Fourier transform on lattice site $\mathbf{R}_j$ with 4-degree of freedom, namely

$$\mathrm{dof} = (e\uparrow, h\uparrow, e\downarrow, h\downarrow)$$

in oder to convert the momentum space hamiltonian to the real space lattice one, we need to consider the following terms

$$
\begin{aligned}
\sum_j c_{j,\sigma}^\dagger c_{j+l,\sigma'}^\dagger &= \frac{1}{N} \sum_j \sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{R}_j \cdot \mathbf{k}} c_{\mathbf{k},\sigma}^\dagger e^{-i\mathbf{R}_{j+l} \cdot \mathbf{k}'} c_{\mathbf{k}',\sigma'}^\dagger \\
&= \frac{1}{N} \sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{R}_l \cdot \mathbf{k}'} \sum_j e^{-i\mathbf{R}_j \cdot (\mathbf{k}+\mathbf{k}')} c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k}',\sigma'}^\dagger \\
&= \sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{R}_l \cdot \mathbf{k}'} \delta_{\mathbf{k},-\mathbf{k}'} c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k}',\sigma'}^\dagger \\
&= \sum_{\mathbf{k}} e^{i\mathbf{R}_l \cdot \mathbf{k}} c_{\mathbf{k},\sigma}^\dagger c_{-\mathbf{k},\sigma'}^\dagger
\end{aligned}
$$

$$
\begin{aligned}
\sum_j c_{j,\sigma}^\dagger c_{j+l,\sigma'} &= \frac{1}{N} \sum_j \sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{R}_j \cdot \mathbf{k}} c_{\mathbf{k},\sigma}^\dagger e^{i\mathbf{R}_{j+l} \cdot \mathbf{k}'} c_{\mathbf{k}',\sigma'} \\
&= \frac{1}{N} \sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{R}_l \cdot \mathbf{k}'} \sum_j e^{-i\mathbf{R}_j \cdot (\mathbf{k}-\mathbf{k}')} c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k}',\sigma'} \\
&= \sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{R}_l \cdot \mathbf{k}'} \delta_{\mathbf{k},\mathbf{k}'} c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k}',\sigma'} \\
&= \sum_{\mathbf{k}} e^{i\mathbf{R}_l \cdot \mathbf{k}} c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k},\sigma'}
\end{aligned}
$$

$$
\sum_j c_{j,\sigma} c_{j+l,\sigma'}^\dagger = \frac{1}{N} \sum_j \sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{R}_j \cdot \mathbf{k}} c_{\mathbf{k},\sigma} e^{-i\mathbf{R}_{j+l} \cdot \mathbf{k}'} c_{\mathbf{k}',\sigma'}^\dagger
$$

$$= \frac{1}{N}\sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{R}_l\cdot\mathbf{k}'}\sum_j e^{i\mathbf{R}_j\cdot(\mathbf{k}-\mathbf{k}')} c_{\mathbf{k},\sigma}c^\dagger_{\mathbf{k}',\sigma'}$$
$$= \sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{R}_l\cdot\mathbf{k}'}\delta_{\mathbf{k},\mathbf{k}'} c_{\mathbf{k},\sigma}c^\dagger_{\mathbf{k}',\sigma'}$$
$$= \sum_{\mathbf{k}} e^{-i\mathbf{R}_l\cdot\mathbf{k}} c_{\mathbf{k},\sigma}c^\dagger_{\mathbf{k},\sigma'}$$

$$\sum_j c_{j,\sigma}c_{j+l,\sigma'} = \frac{1}{N}\sum_j\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{R}_j\cdot\mathbf{k}} c_{\mathbf{k},\sigma} e^{i\mathbf{R}_{j+l}\cdot\mathbf{k}'} c_{\mathbf{k}',\sigma'}$$
$$= \frac{1}{N}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{R}_l\cdot\mathbf{k}'}\sum_j e^{i\mathbf{R}_j\cdot(\mathbf{k}+\mathbf{k}')} c_{\mathbf{k},\sigma}c_{\mathbf{k}',\sigma'}$$
$$= \sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{R}_l\cdot\mathbf{k}'}\delta_{\mathbf{k},-\mathbf{k}'} c_{\mathbf{k},\sigma}c_{\mathbf{k}',\sigma'}$$
$$= \sum_{\mathbf{k}} e^{-i\mathbf{R}_l\cdot\mathbf{k}} c_{\mathbf{k},\sigma}c_{-\mathbf{k},\sigma'}$$

so we have the following

$$\sum_{\mathbf{k}} \cos(\mathbf{R}_l\cdot\mathbf{k}) c^\dagger_{\mathbf{k},\sigma}c^\dagger_{-\mathbf{k},\sigma'} = \frac{1}{2}\sum_j (c^\dagger_{j,\sigma}c^\dagger_{j+l,\sigma'} + c^\dagger_{j,\sigma}c^\dagger_{j-l,\sigma'}) \tag{11}$$
$$\sum_{\mathbf{k}} \sin(\mathbf{R}_l\cdot\mathbf{k}) c^\dagger_{\mathbf{k},\sigma}c^\dagger_{-\mathbf{k},\sigma'} = -\frac{i}{2}\sum_j (c^\dagger_{j,\sigma}c^\dagger_{j+l,\sigma'} - c^\dagger_{j,\sigma}c^\dagger_{j-l,\sigma'}) \tag{12}$$

$$\sum_{\mathbf{k}} \cos(\mathbf{R}_l\cdot\mathbf{k}) c^\dagger_{\mathbf{k},\sigma}c_{\mathbf{k},\sigma'} = \frac{1}{2}\sum_j (c^\dagger_{j,\sigma}c_{j+l,\sigma'} + c^\dagger_{j,\sigma}c_{j-l,\sigma'}) \tag{13}$$
$$\sum_{\mathbf{k}} \sin(\mathbf{R}_l\cdot\mathbf{k}) c^\dagger_{\mathbf{k},\sigma}c_{\mathbf{k},\sigma'} = -\frac{i}{2}\sum_j (c^\dagger_{j,\sigma}c_{j+l,\sigma'} - c^\dagger_{j,\sigma}c_{j-l,\sigma'}) \tag{14}$$

$$\sum_{\mathbf{k}} \cos(\mathbf{R}_l\cdot\mathbf{k}) c_{\mathbf{k},\sigma}c^\dagger_{\mathbf{k},\sigma'} = \frac{1}{2}\sum_j (c_{j,\sigma}c^\dagger_{j+l,\sigma'} + c_{j,\sigma}c^\dagger_{j-l,\sigma'}) \tag{15}$$
$$\sum_{\mathbf{k}} \sin(\mathbf{R}_l\cdot\mathbf{k}) c_{\mathbf{k},\sigma}c^\dagger_{\mathbf{k},\sigma'} = \frac{i}{2}\sum_j (c_{j,\sigma}c^\dagger_{j+l,\sigma'} - c_{j,\sigma}c^\dagger_{j-l,\sigma'}) \tag{16}$$

$$\sum_{\mathbf{k}} \cos(\mathbf{R}_l\cdot\mathbf{k}) c_{\mathbf{k},\sigma}c_{-\mathbf{k},\sigma'} = \frac{1}{2}\sum_j (c_{j,\sigma}c_{j+l,\sigma'} + c_{j,\sigma}c_{j-l,\sigma'}) \tag{17}$$
$$\sum_{\mathbf{k}} \sin(\mathbf{R}_l\cdot\mathbf{k}) c_{\mathbf{k},\sigma}c_{-\mathbf{k},\sigma'} = \frac{i}{2}\sum_j (c_{j,\sigma}c_{j+l,\sigma'} - c_{j,\sigma}c_{j-l,\sigma'}) \tag{18}$$

using the above formula, we find that

$$\sin k_x\sigma_z\otimes s_x = \sin k_x(c^\dagger_{\mathbf{k},\uparrow}c^\dagger_{-\mathbf{k},\uparrow} + c_{-\mathbf{k},\uparrow}c_{\mathbf{k},\uparrow} - c^\dagger_{\mathbf{k},\downarrow}c^\dagger_{-\mathbf{k},\downarrow} - c_{-\mathbf{k},\downarrow}c_{\mathbf{k},\downarrow})$$
$$= -\frac{i}{2}\sum_j (c^\dagger_{j,\uparrow}c^\dagger_{j+x,\uparrow} - c^\dagger_{j,\uparrow}c^\dagger_{j-x,\uparrow}) - \frac{i}{2}\sum_j (c_{j,\uparrow}c_{j+x,\uparrow} - c_{j,\uparrow}c_{j-x,\uparrow})$$
$$+ \frac{i}{2}\sum_j (c^\dagger_{j,\downarrow}c^\dagger_{j+x,\downarrow} - c^\dagger_{j,\downarrow}c^\dagger_{j-x,\downarrow}) + \frac{i}{2}\sum_j (c_{j,\downarrow}c_{j+x,\downarrow} - c_{j,\downarrow}c_{j-x,\downarrow})$$

$$\sin k_y I\otimes s_y = \sin k_y(-ic^\dagger_{\mathbf{k},\uparrow}c^\dagger_{-\mathbf{k},\uparrow} + ic_{-\mathbf{k},\uparrow}c_{\mathbf{k},\uparrow} - ic^\dagger_{\mathbf{k},\downarrow}c^\dagger_{-\mathbf{k},\downarrow} + ic_{-\mathbf{k},\downarrow}c_{\mathbf{k},\downarrow})$$

$$
= -\frac{i}{2}\sum_j -i(c^\dagger_{j,\uparrow}c^\dagger_{j+y,\uparrow} - c^\dagger_{j,\uparrow}c^\dagger_{j-y,\uparrow}) - \frac{i}{2}\sum_j i(c_{j,\uparrow}c_{j+y,\uparrow} - c_{j,\uparrow}c_{j-y,\uparrow})
$$
$$
+ \frac{i}{2}\sum_j i(c^\dagger_{j,\downarrow}c^\dagger_{j+y,\downarrow} - c^\dagger_{j,\downarrow}c^\dagger_{j-y,\downarrow}) + \frac{i}{2}\sum_j -i(c_{j,\downarrow}c_{j+y,\downarrow} - c_{j,\downarrow}c_{j-y,\downarrow})
$$

$$
\sin k_z \sigma_x \otimes s_x = \sin k_z (c^\dagger_{\mathbf{k},\uparrow}c^\dagger_{-\mathbf{k},\downarrow} + c_{-\mathbf{k},\uparrow}c_{\mathbf{k},\downarrow} + c^\dagger_{\mathbf{k},\downarrow}c^\dagger_{-\mathbf{k},\uparrow} + c_{-\mathbf{k},\downarrow}c_{\mathbf{k},\uparrow})
$$
$$
= -\frac{i}{2}\sum_j (c^\dagger_{j,\uparrow}c^\dagger_{j+z,\downarrow} - c^\dagger_{j,\uparrow}c^\dagger_{j-z,\downarrow}) - \frac{i}{2}\sum_j (c_{j,\uparrow}c_{j+z,\downarrow} - c_{j,\uparrow}c_{j-z,\downarrow})
$$
$$
+ \frac{i}{2}\sum_j -(c^\dagger_{j,\downarrow}c^\dagger_{j+z,\uparrow} - c^\dagger_{j,\downarrow}c^\dagger_{j-z,\uparrow}) + \frac{i}{2}\sum_j -(c_{j,\downarrow}c_{j+z,\uparrow} - c_{j,\downarrow}c_{j-z,\uparrow})
$$

$$
2(\cos k_x + \cos k_y + \cos k_z) I \otimes s_z = 2\sum_\delta \cos k_\delta (c^\dagger_{\mathbf{k},\uparrow}c_{\mathbf{k},\uparrow} - c_{-\mathbf{k},\uparrow}c^\dagger_{-\mathbf{k},\uparrow} + c^\dagger_{\mathbf{k},\downarrow}c_{\mathbf{k},\downarrow} - c_{-\mathbf{k},\downarrow}c^\dagger_{-\mathbf{k},\downarrow})
$$
$$
= 2\sum_\delta \{\frac{1}{2}\sum_j (c^\dagger_{j,\uparrow}c_{j+\delta,\uparrow} + c^\dagger_{j,\uparrow}c_{j-\delta,\uparrow}) - \frac{1}{2}\sum_j (c_{j,\uparrow}c^\dagger_{j+\delta,\uparrow} + c_{j,\uparrow}c^\dagger_{j-\delta,\uparrow})
$$
$$
+ \frac{1}{2}\sum_j (c^\dagger_{j,\downarrow}c_{j+\delta,\downarrow} + c^\dagger_{j,\downarrow}c_{j-\delta,\downarrow}) - \frac{1}{2}\sum_j (c_{j,\downarrow}c^\dagger_{j+\delta,\downarrow} + c_{j,\downarrow}c^\dagger_{j-\delta,\downarrow})\}
$$

using the commutation relations $\{c_{j,\sigma}, c_{j+\delta,\sigma'}\} = 0$, $\{c^\dagger_{j,\sigma}, c^\dagger_{j+\delta,\sigma'}\} = 0$ etc, we can simplifying the above real space lattice hamiltonian to

$$
\Delta \sin k_x \sigma_z \otimes s_x = \Delta \sum_j (ic^\dagger_{j+x,\uparrow}c^\dagger_{j,\uparrow} - ic_{j,\uparrow}c_{j+x,\uparrow} - ic^\dagger_{j+x,\downarrow}c^\dagger_{j,\downarrow} + ic_{j,\downarrow}c_{j+x,\downarrow})
$$

$$
\Delta \sin k_y I \otimes s_y = \Delta \sum_j (c^\dagger_{j+y,\uparrow}c^\dagger_{j,\uparrow} + c_{j,\uparrow}c_{j+y,\uparrow} + c^\dagger_{j+y,\downarrow}c^\dagger_{j,\downarrow} + c_{j,\downarrow}c_{j+y,\downarrow})
$$

$$
\Delta \sin k_z \sigma_x \otimes s_x = \Delta \sum_j (ic^\dagger_{j+z,\downarrow}c^\dagger_{j,\uparrow} + ic^\dagger_{j+z,\uparrow}c^\dagger_{j,\downarrow} - ic_{j,\uparrow}c_{j+z,\downarrow} - ic_{j,\downarrow}c_{j+z,\uparrow})
$$

$$
2(\cos k_x + \cos k_y + \cos k_z) I \otimes s_z = 2\sum_{\delta,j} (c^\dagger_{j+\delta,\uparrow}c_{j,\uparrow} + c^\dagger_{j,\uparrow}c_{j+\delta,\uparrow} + c^\dagger_{j+\delta,\downarrow}c_{j,\downarrow} + c^\dagger_{j,\downarrow}c_{j+\delta,\downarrow})
$$
$$
= 2\sum_{\delta,j,\sigma} (c^\dagger_{j+\delta,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{j+\delta,\sigma})
$$

$$
I \otimes s_z = \sum_j (c^\dagger_{j,\uparrow}c_{j,\uparrow} - c_{j,\uparrow}c^\dagger_{j,\uparrow} + c^\dagger_{j,\downarrow}c_{j,\downarrow} - c_{j,\downarrow}c^\dagger_{j,\downarrow})
$$

so the real space lattice hamiltonian for this model becomes

$$
H(k) = \Delta \sin k_x s_x \otimes \sigma_z + \Delta \sin k_y s_y \otimes I + (-\Delta \sin k_z) s_x \otimes \sigma_x
$$
$$
+ \{2t(\cos k_x + \cos k_y + \cos k_z) - \mu\} s_z \otimes I
$$
$$
\rightarrow H = \Delta \sum_j (ic^\dagger_{j+x,\uparrow}c^\dagger_{j,\uparrow} - ic_{j,\uparrow}c_{j+x,\uparrow} - ic^\dagger_{j+x,\downarrow}c^\dagger_{j,\downarrow} + ic_{j,\downarrow}c_{j+x,\downarrow})
$$
$$
+ \Delta \sum_j (c^\dagger_{j+y,\uparrow}c^\dagger_{j,\uparrow} + c_{j,\uparrow}c_{j+y,\uparrow} + c^\dagger_{j+y,\downarrow}c^\dagger_{j,\downarrow} + c_{j,\downarrow}c_{j+y,\downarrow})
$$

$$
\begin{aligned}
&-\Delta\sum_j(ic^\dagger_{j+z,\downarrow}c^\dagger_{j,\uparrow}+ic^\dagger_{j+z,\uparrow}c^\dagger_{j,\downarrow}-ic_{j,\uparrow}c_{j+z,\downarrow}-ic_{j,\downarrow}c_{j+z,\uparrow})\\
&+2t\sum_{\delta,j,\sigma}(c^\dagger_{j+\delta,\sigma}c_{j,\sigma}+c^\dagger_{j,\sigma}c_{j+\delta,\sigma})\\
&-\mu\sum_j(c^\dagger_{j,\uparrow}c_{j,\uparrow}-c_{j,\uparrow}c^\dagger_{j,\uparrow}+c^\dagger_{j,\downarrow}c_{j,\downarrow}-c_{j,\downarrow}c^\dagger_{j,\downarrow})
\end{aligned}
$$

if we write the lattice tight binding hamiltonian in terms of the coupling vector and matrices in the basis

$$\text{dof}=(e\uparrow,h\uparrow,e\downarrow,h\downarrow)$$

then they are

$$[-1,0,0]\leftrightarrow\begin{pmatrix}0&i\Delta&0&0\\0&0&0&0\\0&0&0&-i\Delta\\0&0&0&0\end{pmatrix}\quad[1,0,0]\leftrightarrow\begin{pmatrix}0&0&0&0\\-i\Delta&0&0&0\\0&0&0&0\\0&0&i\Delta&0\end{pmatrix}$$

$$[0,-1,0]\leftrightarrow\begin{pmatrix}0&\Delta&0&0\\0&0&0&0\\0&0&0&\Delta\\0&0&0&0\end{pmatrix}\quad[0,1,0]\leftrightarrow\begin{pmatrix}0&0&0&0\\\Delta&0&0&0\\0&0&0&0\\0&0&\Delta&0\end{pmatrix}$$

$$[0,0,-1]\leftrightarrow\begin{pmatrix}0&0&0&-i\Delta\\0&0&0&0\\0&-i\Delta&0&0\\0&0&0&0\end{pmatrix}\quad[0,0,1]\leftrightarrow\begin{pmatrix}0&0&0&0\\0&0&i\Delta&0\\0&0&0&0\\i\Delta&0&0&0\end{pmatrix}$$

$$[0,0,0]\leftrightarrow\begin{pmatrix}-\mu&0&0&0\\0&\mu&0&0\\0&0&-\mu&0\\0&0&0&\mu\end{pmatrix}$$

$$-\delta\leftrightarrow\begin{pmatrix}t&0&0&0\\0&-t&0&0\\0&0&t&0\\0&0&0&-t\end{pmatrix}\quad\delta\leftrightarrow\begin{pmatrix}t&0&0&0\\0&-t&0&0\\0&0&t&0\\0&0&0&-t\end{pmatrix}$$

where we have written the term parametrized by t in a more symmetric one

$$2t\sum_{\delta,j,\sigma}(c^\dagger_{j+\delta,\sigma}c_{j,\sigma}+c^\dagger_{j,\sigma}c_{j+\delta,\sigma})=\sum_{\delta,j,\sigma}(tc^\dagger_{j+\delta,\sigma}c_{j,\sigma}+tc^\dagger_{j,\sigma}c_{j+\delta,\sigma}-tc_{j,\sigma}c^\dagger_{j+\delta,\sigma}-tc_{j+\delta,\sigma}c^\dagger_{j,\sigma})$$

similarly, in this case, n=2 and W is the chiral operator $\Gamma_5=s_x\otimes\sigma_y$, and

$$2^ni^n\to\text{tr}[\Gamma_5\Gamma_3\Gamma_1\Gamma_2\Gamma_4]=-2^2$$

so the topological operator can be written as

$$\hat{C}_{3D-DIII}=i\frac{32\pi}{2^ni^n}W(QXPYQZP+PXQYPZQ)=-8\pi i\Gamma_5(QXPYQZP+PXQYPZQ)$$

∮.3 3D-AII-prototype Topological Insulators, Bi_2Se_3 and Bi_2Te_3 etc.

The 3D class AII is relevant to prototype TIs such as Bi2Se3 and Bi2Te3, the Dirac matrices used in the low energy effective hamiltonian is given by[10][11]

$$\Gamma_{1\sim 5} = \{\sigma_x \otimes \tau_x, \sigma_y \otimes \tau_x, \sigma_z \otimes \tau_x, I \otimes \tau_y, I \otimes \tau_z\}$$

in this case, there is still 3 real matrices and 2 purely imaginary matrices. and the operator B is given by

$$B = (\sigma_x \otimes \tau_x)(\sigma_z \otimes \tau_x)(I \otimes \tau_z) = -i\sigma_y \otimes \tau_z$$

and we choose $\gamma_0 = I \otimes \tau_z$ and have $A = B\gamma_0 = -i\sigma_y \otimes I$ which is the time reversal operator squares to -1. since 3D AII is in the first descendant series, we should omitted some factor $d_i(\mathbf{k})$ to derive the 3D AII class Dirac hamiltonian. we choose Γ_3 to be omitted, and consider the following Dirac hamiltonian

$$H(k) = (M + M_1 k_z^2 + M_2 k_x^2 + M_2 k_y^2)(I \otimes \tau_z) + B_0 k_z I \otimes \tau_y + A_0 k_y \sigma_x \otimes \tau_x - A_0 k_x \sigma_y \otimes \tau_x \tag{19}$$

following the same strategy above, we $k \to \sin k, k^2 \to 2(1-\cos k)$,this model can be written as

$$\begin{aligned} H(k) = &(M + 2M_1 + 4M_2 - 2M_1\cos k_z - 2M_2\cos k_x - 2M_2\cos k_y)(I \otimes \tau_z) \\ &+ B_0 \sin k_z I \otimes \tau_y + A_0 \sin k_y \sigma_x \otimes \tau_x - A_0 \sin k_x \sigma_y \otimes \tau_x \end{aligned}$$

in order to derive the real space lattice hamiltonian, we can consider the spinor in this case is written as $\psi_{\mathbf{k}} = (c_{\mathbf{k}s,\uparrow}, c_{\mathbf{k}p,\uparrow}, c_{\mathbf{k}s,\downarrow}, c_{\mathbf{k}p,\downarrow})^T$, where s and p is the orbital degree of freedom which τ acting on. following the same strategy as that discussed in the 3D class AIII, we can find that

$$\begin{aligned} c_{\mathbf{k},\alpha,\sigma} &= \frac{1}{\sqrt{N}} \sum_j e^{-i\mathbf{R}_j\cdot\mathbf{k}} c_{j,\alpha,\sigma} \\ c_{j,\alpha,\sigma} &= \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{R}_j\cdot\mathbf{k}} c_{\mathbf{k},\alpha,\sigma} \end{aligned}$$

$$\begin{aligned} \sum_j c^\dagger_{j,\alpha,\sigma} c_{j+l,\alpha',\sigma'} &= \sum_k e^{i\mathbf{R}_l\cdot\mathbf{k}} c^\dagger_{\mathbf{k},\alpha,\sigma} c_{\mathbf{k},\alpha',\sigma'} \\ \to \sum_k \cos(\mathbf{R}_l\cdot\mathbf{k}) c^\dagger_{\mathbf{k},\alpha,\sigma} c_{\mathbf{k},\alpha',\sigma'} &= \frac{1}{2}(\sum_j c^\dagger_{j,\alpha,\sigma} c_{j+l,\alpha',\sigma'} + \sum_j c^\dagger_{j,\alpha,\sigma} c_{j-l,\alpha',\sigma'}) \\ \to \sum_k \sin(\mathbf{R}_l\cdot\mathbf{k}) c^\dagger_{\mathbf{k},\alpha,\sigma} c_{\mathbf{k},\alpha',\sigma'} &= -\frac{i}{2}(\sum_j c^\dagger_{j,\alpha,\sigma} c_{j+l,\alpha',\sigma'} - \sum_j c^\dagger_{j,\alpha,\sigma} c_{j-l,\alpha',\sigma'}) \end{aligned}$$

then we have the real space lattice hamiltonian associated with the above Dirac hamiltonian

$$\begin{aligned} &(M + 2M_1 + 4M_2) I \otimes \tau_z = (M + 2M_1 + 4M_2)(c^\dagger_{\mathbf{k},s,\uparrow} c_{\mathbf{k},s,\uparrow} - c^\dagger_{\mathbf{k},p,\uparrow} c_{\mathbf{k},p,\uparrow} + c^\dagger_{\mathbf{k},s,\downarrow} c_{\mathbf{k},s,\downarrow} - c^\dagger_{\mathbf{k},p,\downarrow} c_{\mathbf{k},p,\downarrow} \\ &= (M + 2M_1 + 4M_2) \sum_j (c^\dagger_{j,s,\uparrow} c_{j,s,\uparrow} - c^\dagger_{j,p,\uparrow} c_{j,p,\uparrow} + c^\dagger_{j,s,\downarrow} c_{j,s,\downarrow} - c^\dagger_{j,p,\downarrow} c_{j,p,\downarrow}) \end{aligned}$$

$$\begin{aligned} \cos k_\delta I \otimes \tau_z &= \cos k_\delta (c^\dagger_{\mathbf{k},s,\uparrow} c_{\mathbf{k},s,\uparrow} - c^\dagger_{\mathbf{k},p,\uparrow} c_{\mathbf{k},p,\uparrow} + c^\dagger_{\mathbf{k},s,\downarrow} c_{\mathbf{k},s,\downarrow} - c^\dagger_{\mathbf{k},p,\downarrow} c_{\mathbf{k},p,\downarrow} \\ &= \frac{1}{2} \sum_j \{(c^\dagger_{j,s,\uparrow} c_{j+\delta,s,\uparrow} + c^\dagger_{j,s,\uparrow} c_{j-\delta,s,\uparrow}) - (c^\dagger_{j,p,\uparrow} c_{j+\delta,p,\uparrow} + c^\dagger_{j,p,\uparrow} c_{j-\delta,p,\uparrow}) \end{aligned}$$

$$+(c^\dagger_{j,s,\downarrow}c_{j+\delta,s,\downarrow}+c^\dagger_{j,s,\downarrow}c_{j-\delta,s,\downarrow})-(c^\dagger_{j,p,\downarrow}c_{j+\delta,p,\downarrow}+c^\dagger_{j,p,\downarrow}c_{j-\delta,p,\downarrow})\}$$

$$\sin k_z I\otimes\tau_y=\sin k_z(-ic^\dagger_{\mathbf{k},s,\uparrow}c_{\mathbf{k},p,\uparrow}+ic^\dagger_{\mathbf{k},p,\uparrow}c_{\mathbf{k},s,\uparrow}-ic^\dagger_{\mathbf{k},s,\downarrow}c_{\mathbf{k},p,\downarrow}+ic^\dagger_{\mathbf{k},p,\downarrow}c_{\mathbf{k},s,\downarrow})$$
$$=\frac{1}{2}\sum_j\{-(c^\dagger_{j,s,\uparrow}c_{j+z,p,\uparrow}-c^\dagger_{j,s,\uparrow}c_{j-z,p,\uparrow})+(c^\dagger_{j,p,\uparrow}c_{j+z,s,\uparrow}-c^\dagger_{j,p,\uparrow}c_{j-z,s,\uparrow})$$
$$-(c^\dagger_{j,s,\downarrow}c_{j+z,p,\downarrow}-c^\dagger_{j,s,\downarrow}c_{j-z,p,\downarrow})+(c^\dagger_{j,p,\downarrow}c_{j+z,s,\downarrow}-c^\dagger_{j,p,\downarrow}c_{j-z,s,\downarrow})\}$$

$$\sin k_y\sigma_x\otimes\tau_x=\sin k_y(c^\dagger_{\mathbf{k},s,\uparrow}c_{\mathbf{k},p,\downarrow}+c^\dagger_{\mathbf{k},p,\uparrow}c_{\mathbf{k},s,\downarrow}+c^\dagger_{\mathbf{k},s,\downarrow}c_{\mathbf{k},p,\uparrow}+c^\dagger_{\mathbf{k},p,\downarrow}c_{\mathbf{k},s,\uparrow})$$
$$=-\frac{i}{2}\sum_j\{(c^\dagger_{j,s,\uparrow}c_{j+y,p,\downarrow}-c^\dagger_{j,s,\uparrow}c_{j-y,p,\downarrow})+(c^\dagger_{j,p,\uparrow}c_{j+y,s,\downarrow}-c^\dagger_{j,p,\uparrow}c_{j-y,s,\downarrow})$$
$$(c^\dagger_{j,s,\downarrow}c_{j+y,p,\uparrow}-c^\dagger_{j,s,\downarrow}c_{j-y,p,\uparrow})+(c^\dagger_{j,p,\downarrow}c_{j+y,s,\uparrow}-c^\dagger_{j,p,\downarrow}c_{j-y,s,\uparrow})\}$$

$$\sin k_x\sigma_y\otimes\tau_x=\sin k_x(-ic^\dagger_{\mathbf{k},s,\uparrow}c_{\mathbf{k},p,\downarrow}-ic^\dagger_{\mathbf{k},p,\uparrow}c_{\mathbf{k},s,\downarrow}+ic^\dagger_{\mathbf{k},s,\downarrow}c_{\mathbf{k},p,\uparrow}+ic^\dagger_{\mathbf{k},p,\downarrow}c_{\mathbf{k},s,\uparrow})$$
$$=\frac{1}{2}\sum_j\{-(c^\dagger_{j,s,\uparrow}c_{j+x,p,\downarrow}-c^\dagger_{j,s,\uparrow}c_{j-x,p,\downarrow})-(c^\dagger_{j,p,\uparrow}c_{j+x,s,\downarrow}-c^\dagger_{j,p,\uparrow}c_{j-x,s,\downarrow})$$
$$(c^\dagger_{j,s,\downarrow}c_{j+x,p,\uparrow}-c^\dagger_{j,s,\downarrow}c_{j-x,p,\uparrow})+(c^\dagger_{j,p,\downarrow}c_{j+x,s,\uparrow}-c^\dagger_{j,p,\downarrow}c_{j-x,s,\uparrow})\}$$

considering the extra chemical potential term $-\mu\sum_{j,\alpha,\sigma}c^\dagger_{j,\alpha,\sigma}c_{j,\alpha,\sigma}$, the real space lattice hamiltonian can be written as[12]

$$H=-\mu\sum_{j,\alpha,\sigma}c^\dagger_{j,\alpha,\sigma}c_{j,\alpha,\sigma}+(M+2M_1+4M_2)\sum_j(c^\dagger_{j,s,\uparrow}c_{j,s,\uparrow}-c^\dagger_{j,p,\uparrow}c_{j,p,\uparrow}+c^\dagger_{j,s,\downarrow}c_{j,s,\downarrow}-c^\dagger_{j,p,\downarrow}c_{j,p,\downarrow})$$
$$-\frac{A_0}{2}\sum_j\{-(c^\dagger_{j,s,\uparrow}c_{j+x,p,\downarrow}-c^\dagger_{j+x,s,\uparrow}c_{j,p,\downarrow})-(c^\dagger_{j,p,\uparrow}c_{j+x,s,\downarrow}-c^\dagger_{j+x,p,\uparrow}c_{j,s,\downarrow})\}+H.c$$
$$-i\frac{A_0}{2}\sum_j\{(c^\dagger_{j,s,\uparrow}c_{j+y,p,\downarrow}-c^\dagger_{j+y,s,\uparrow}c_{j,p,\downarrow})+(c^\dagger_{j,p,\uparrow}c_{j+y,s,\downarrow}-c^\dagger_{j+y,p,\uparrow}c_{j,s,\downarrow})\}+H.c$$
$$+\frac{B_0}{2}\sum_j\{-(c^\dagger_{j,s,\uparrow}c_{j+z,p,\uparrow}-c^\dagger_{j+z,s,\uparrow}c_{j,p,\uparrow})-(c^\dagger_{j,s,\downarrow}c_{j+z,p,\downarrow}-c^\dagger_{j+z,s,\downarrow}c_{j,p,\downarrow})\}+H.c$$
$$-M_1\sum_{j,\delta=z}\{c^\dagger_{j,s,\uparrow}c_{j+\delta,s,\uparrow}-c^\dagger_{j,p,\uparrow}c_{j+\delta,p,\uparrow}+c^\dagger_{j,s,\downarrow}c_{j+\delta,s,\downarrow}-c^\dagger_{j,p,\downarrow}c_{j+\delta,p,\downarrow}\}+H.c$$
$$-M_2\sum_{j,\delta=x,y}\{c^\dagger_{j,s,\uparrow}c_{j+\delta,s,\uparrow}-c^\dagger_{j,p,\uparrow}c_{j+\delta,p,\uparrow}+c^\dagger_{j,s,\downarrow}c_{j+\delta,s,\downarrow}-c^\dagger_{j,p,\downarrow}c_{j+\delta,p,\downarrow}\}+H.c$$

in this class, n is also equal to 2 and W is equal to $W=\Gamma_3=\sigma_z\otimes\tau_x$, and

$$2^n i^n\to\mathrm{tr}[\Gamma_3\Gamma_5\Gamma_2\Gamma_1\Gamma_4]=-2^2$$

so the topological operator is

$$\hat{C}_{3D-AII}=i\frac{32\pi}{2^n i^n}W(QXPYQZP+PXQYPZQ)=-8\pi i\Gamma_3(QXPYQZP+PXQYPZQ)$$

∮.4 3D-CII

as for the 3D CII, since it's in the second descendant series, we should construct the Dirac matrices for the the primary series with the same symmetry, which is the 5D symmetry class CII, the Clifford algebra is Cl^7, the Dirac metrics is at least $2^3 = 8$ dimensional. we can construct it by induction from the Clifford algebra Cl^5 used in the previous discussion in 3D class AIII. that is

$$\gamma_1 = \tau_x \otimes \sigma_x \otimes \eta_x$$
$$\gamma_2 = \tau_x \otimes \sigma_y \otimes \eta_x$$
$$\gamma_3 = \tau_x \otimes \sigma_z \otimes \eta_x$$
$$\gamma_4 = \tau_z \otimes I \otimes \eta_x$$
$$\gamma_5 = \tau_y \otimes I \otimes \eta_x$$
$$\gamma_6 = I \otimes I \otimes \eta_y$$
$$\gamma_7 = I \otimes I \otimes \eta_z$$

where the previous five matrices form the Clifford algebra Cl^5 which has been used in 3D class AIII. in order to make it as to that used in literature[13],[5], we can re-ordering these matrices, that is:

$$\Gamma_1 = \gamma_1 = \tau_x \otimes \sigma_x \otimes \eta_x$$
$$\Gamma_2 = \gamma_2 = \tau_x \otimes \sigma_y \otimes \eta_x$$
$$\Gamma_3 = \gamma_3 = \tau_x \otimes \sigma_z \otimes \eta_x$$
$$\Gamma_4 = \gamma_4 = \tau_z \otimes I \otimes \eta_x$$
$$-\Gamma_7 = \gamma_5 = \tau_y \otimes I \otimes \eta_x$$
$$\Gamma_5 = \gamma_6 = I \otimes I \otimes \eta_y$$
$$\Gamma_6 = \gamma_7 = I \otimes I \otimes \eta_z$$

the product of all the real matrices is

$$B = -\tau_z \otimes \sigma_y \otimes \eta_y$$

the chiral operator is chosen as $S = \Gamma_6 = I \otimes I \otimes \eta_z$, and the matrix Γ_1 is chosen to be the role of Γ_{D+1}, thus, we can find that

$$A = B\Gamma_{D+1} = i\tau_y \otimes \sigma_z \otimes \eta_z$$

which serve as the particle hole operator which squares to -1 for the Dirac hamiltonian

$$H(k) = \sum_{i=1,2,3,4,5,7} d_i \Gamma_i$$

and $AS = i\tau_y \otimes \sigma_z \otimes I$ serve as the time reversal operator which squares to -1.

the real space lattice hamiltonian are less considering in the literature for this type of Dirac hamiltonian, so we ignore the real space lattice version of this one at present.

since CII is in the second descendant, we should set two $d_i(k)$ to be zero so as to use the dimensional reduction, we choose $d_5 = d_7 = 0$, thus the omitted matrices are $\Gamma_5, \Gamma_6, \Gamma_7$ and we have $n = [\frac{D+2+1}{2}] = 3$ and $W = \Gamma_5, \Gamma_6, \Gamma_7$, so the topological operator is

$$\hat{C}_{3D-CII} = \frac{32\pi}{2^n i^n} W(QXPYQZP + PXQYPZQ) = -4\pi i \Gamma_5\Gamma_6\Gamma_7(QXPYQZP + PXQYPZQ)$$

∮.5 3D-CI

since in 3D, symmetry class CI lies in the even series, we should consider the Clifford algebra $Cl^{2n+1} = Cl^7$ with $n = \frac{D+3}{2} = 3$, which is the same as that discussed in the class CII, we can pick up a Γ_i as the chiral operator, namely $S = \Gamma_6 = I \otimes I \otimes \eta_z$. in this case, we can choose $\Gamma_0 = -i\Gamma_7$ and construct the Dirac hamiltonian as[13]

$$H(k) = \sum_{i=0,1,2,3} d_i \Gamma_i$$

similarly, in this case the product of all the real matrices is

$$B = -\tau_z \otimes \sigma_y \otimes \eta_y$$

serve as the time reversal operator which squares to +1 and

$$BS = -\tau_z \otimes \sigma_y \otimes i\eta_x = -\tau_z \otimes i\sigma_y \otimes \eta_x$$

serve as the particle-hole operator which squares to -1.

since the real space lattice model of this type hamiltonian has been less discussed in the literature due to the complexity of the 8 dimensional Dirac matrices, we ignore the real space lattice model for this case at present.

since we only use $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_7$, so $W = \Gamma_4\Gamma_5, \Gamma_6$, similarly, in this case $n = [\frac{D+3}{2}] = 3$, so, we have the topological operator can be written as

$$\hat{C}_{3D-CI} = \frac{32\pi}{2^n i^n} W(QXPYQZP + PXQYPZQ) = -4\pi i \Gamma_4\Gamma_5\Gamma_6(QXPYQZP + PXQYPZQ)$$

$\Re$.3 Explicit form for different symmetry classes in two dimension

$\oint$.1 2D-A-The Integer quantum hall effect

in the 2D, typical symmetry class A system is the well know quantum hall effect system, the Clifford algebra for this case is spanned by the three Pauli matrices, and the Dirac hamiltonian can be written as[12](at present we follow the convention used by the author of the article[2])

$$H(k) = A\sin k_x \sigma_x + A\sin k_y \sigma_y + (M + 4B - 2B\cos k_x - 2B\cos k_y)\sigma_z$$

and the spinor can be written as $\psi_{\mathbf{k}} = (c_{\mathbf{k},s}, c_{\mathbf{k},p})^T$, where s and p donate the two degree of freedom in the unite cell.using the sam strategy discussed in the 3D AIII class, we have:

$$\sum_{\mathbf{k}} \sin k_l c^\dagger_{\mathbf{k},I} c_{\mathbf{k},J} = \frac{i}{2}\sum_i (c^\dagger_{i+X_l,I} c_{i,J} - c^\dagger_{i-X_l,I} c_{i,J}) = \frac{i}{2}\sum_i (c^\dagger_{i+X_l,I} c_{i,J} - c^\dagger_{i,I} c_{i+X_l,J})$$
$$\sum_{\mathbf{k}} \cos k_l c^\dagger_{\mathbf{k},I} c_{\mathbf{k},J} = \frac{1}{2}\sum_i (c^\dagger_{i+X_l,I} c_{i,J} + c^\dagger_{i-X_l,I} c_{i,J}) = \frac{1}{2}\sum_i (c^\dagger_{i+X_l,I} c_{i,J} + c^\dagger_{i,I} c_{i+X_l,J})$$

we have the real space lattice version of these terms

$$\sin k_x \sigma_x = \sin k_x (c^\dagger_{\mathbf{k},s} c_{\mathbf{k},p} + c^\dagger_{\mathbf{k},p} c_{\mathbf{k},s})$$
$$= \frac{i}{2}\sum_j \{(c^\dagger_{j+x,s} c_{j,p} - c^\dagger_{j,s} c_{j+x,p}) + (c^\dagger_{j+x,p} c_{j,s} - c^\dagger_{j,p} c_{j+x,s})\}$$

$$\sin k_y \sigma_y = \sin k_y (-ic^\dagger_{\mathbf{k},s} c_{\mathbf{k},p} + ic^\dagger_{\mathbf{k},p} c_{\mathbf{k},s})$$
$$= \frac{i}{2}\sum_j \{-i(c^\dagger_{j+y,s} c_{j,p} - c^\dagger_{j,s} c_{j+y,p}) + i(c^\dagger_{j+y,p} c_{j,s} - c^\dagger_{j,p} c_{j+y,s})\}$$

$$\begin{aligned}\sigma_z &= (c^\dagger_{\mathbf{k},s}c_{\mathbf{k},s} - c^\dagger_{\mathbf{k},p}c_{\mathbf{k},p}) \\ &= \sum_j (c^\dagger_{j,s}c_{j,s} - c^\dagger_{j,p}c_{j,p})\end{aligned}$$

$$\begin{aligned}&\cos k_x\sigma_z = \cos k_x(c^\dagger_{\mathbf{k},s}c_{\mathbf{k},s} - c^\dagger_{\mathbf{k},p}c_{\mathbf{k},p}) \\ &= \frac{1}{2}\sum_j\{(c^\dagger_{j+x,s}c_{j,s} + c^\dagger_{j,s}c_{j+x,s}) - (c^\dagger_{j+x,p}c_{j,p} + c^\dagger_{j,p}c_{j+x,p})\}\end{aligned}$$

$$\begin{aligned}&\cos k_y\sigma_z = \cos k_y(c^\dagger_{\mathbf{k},s}c_{\mathbf{k},s} - c^\dagger_{\mathbf{k},p}c_{\mathbf{k},p}) \\ &= \frac{1}{2}\sum_j\{(c^\dagger_{j+y,s}c_{j,s} + c^\dagger_{j,s}c_{j+y,s}) - (c^\dagger_{j+y,p}c_{j,p} + c^\dagger_{j,p}c_{j+y,p})\}\end{aligned}$$

finally, the real space lattice hamiltonian can be written as:[14][12]

$$\begin{aligned}&H = (M+4B)\sum_j (c^\dagger_{j,s}c_{j,s} - c^\dagger_{j,p}c_{j,p}) - \mu\sum_j (c^\dagger_{j,s}c_{j,s} + c^\dagger_{j,p}c_{j,p}) \\ &+ A\frac{i}{2}\sum_j (c^\dagger_{j+x,s}c_{j,p} - c^\dagger_{j,s}c_{j+x,p}) + H.c \\ &+ A\frac{1}{2}\sum_j (c^\dagger_{j+y,s}c_{j,p} - c^\dagger_{j,s}c_{j+y,p}) + H.c \\ &- 2B\frac{1}{2}\sum_j (c^\dagger_{j+x,s}c_{j,s} - c^\dagger_{j+x,p}c_{j,p}) + H.c \\ &- 2B\frac{1}{2}\sum_j (c^\dagger_{j+y,s}c_{j,s} - c^\dagger_{j+y,p}c_{j,p}) + H.c\end{aligned}$$

where we have added the chemical potential term parameterized by μ.

in this case, we have $n = [\frac{D+1}{2}] = 1$, and W=I since all the Pauli matrices are used and we have

$$2^n i^n \rightarrow \mathrm{tr}[\sigma_z\sigma_x\sigma_y] = 2i$$

so the topological operator is

$$\hat{C}_{2D-A} = \frac{4\pi}{2^n i^n}W(QXPYQ - PXQYP) = -2\pi i(QXPYQ - PXQYP)$$

∮.2 2D-D-Spin-less chiral p-wave topological SCs

A concrete system that realizes the 2D class D is the spin-less chiral p-wave SC[13]. in this case, the Clifford algebra is also spanned by the Dirac matrices, and the spinor is spanned by the Nambu spinor without spin, namely $\psi_k = (c_\mathbf{k}, c^\dagger_{-\mathbf{k}})^T$, where the $c_\mathbf{k}$ represent creation of electron-like quasi-particle with momentum $\mathbf{k}$ and $c^\dagger_{-\mathbf{k}}$ represent the creation of hole-like quasi-particle with momentum $-\mathbf{k}$. the real space lattice hamiltonian for this system is described by

$$\begin{aligned}&H = \sum_{j,\delta} t(c^\dagger_j c_{j+\delta} + c^\dagger_{j+\delta}c_j) - \mu\sum_j c^\dagger_j c_j \\ &+ \sum_j \Delta(-i(c_jc_{j+x} - c_jc_{j-x}) + i(c^\dagger_{j+x}c^\dagger_j - c^\dagger_{j-x}c^\dagger_j) + (c_jc_{j+y} - c_jc_{j+y}) + (c^\dagger_{j+y}c_j - c^\dagger_{j-y}c_j))\end{aligned}$$

using the Fourier Transform

$$c_{\mathbf{k}} = \frac{1}{\sqrt{N}}\sum_j e^{-i\mathbf{R}_j\cdot\mathbf{k}}c_j$$
$$c_{\mathbf{k}}^\dagger = \frac{1}{\sqrt{N}}\sum_j e^{i\mathbf{R}_j\cdot\mathbf{k}}c_j^\dagger$$
$$c_j = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i\mathbf{R}_j\cdot\mathbf{k}}c_{\mathbf{k}}$$
$$c_j^\dagger = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{-i\mathbf{R}_j\cdot\mathbf{k}}c_{\mathbf{k}}^\dagger$$

we can derive the Dirac model for this real space hamiltonian

$$\sum_{j,\delta} t(c_j^\dagger c_{j+\delta} + c_{j+\delta}^\dagger c_j) = \sum_{j,\delta}\sum_{\mathbf{k},\mathbf{k}'}\frac{1}{N}t(e^{-i\mathbf{R}_j\cdot\mathbf{k}}c_{\mathbf{k}}^\dagger e^{i\mathbf{R}_{j+\delta}\cdot\mathbf{k}'}c_{\mathbf{k}'} + e^{-i\mathbf{R}_{j+\delta}\cdot\mathbf{k}}c_{\mathbf{k}}^\dagger e^{i\mathbf{R}_j\cdot\mathbf{k}'}c_{\mathbf{k}'})$$
$$= \sum_{\mathbf{k},\delta} 2t\cos(\mathbf{k}\cdot\mathbf{R}_\delta)c_{\mathbf{k}}^\dagger c_{\mathbf{k}} = \sum_{\mathbf{k},\delta} t\cos(\mathbf{k}\cdot\mathbf{R}_\delta)c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \sum_{\mathbf{k},\delta} t\cos(-\mathbf{k}\cdot\mathbf{R}_\delta)c_{-\mathbf{k}}^\dagger c_{-\mathbf{k}}$$

$$\mu\sum_j c_j^\dagger c_j = \mu\sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} = \frac{\mu}{2}\sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \frac{\mu}{2}\sum_{\mathbf{k}} c_{-\mathbf{k}}^\dagger c_{-\mathbf{k}}$$

$$\sum_j c_j c_{j+\delta} = \frac{1}{N}\sum_j\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{R}_j\cdot\mathbf{k}}c_{\mathbf{k}}e^{i\mathbf{R}_{j+\delta}\cdot\mathbf{k}'}c_{\mathbf{k}'}$$
$$= \sum_{\mathbf{k}} e^{-i\mathbf{R}_\delta\cdot\mathbf{k}}c_{\mathbf{k}}c_{-\mathbf{k}}$$
$$= \sum_{\mathbf{k}} -i\sin(\mathbf{R}_\delta\cdot\mathbf{k})c_{\mathbf{k}}c_{-\mathbf{k}}$$

since $c_{\mathbf{k}}c_{-\mathbf{k}}$ is odd in $\mathbf{k}$, the non-vanishing term must be also odd one in $e^{-i\mathbf{R}_\delta\cdot\mathbf{k}}$ in the total hamiltonian, but as for the specific block label by $\mathbf{k}$, the elements is still $e^{-i\mathbf{R}_\delta\cdot\mathbf{k}}$, so we need to add the term $-\sum_j c_j c_{j+\delta}$ to derive the odd coefficient, namely $2\sin k_\delta$(notice the Nambu spinor we have chosen)

$$\sum_j c_{j+\delta}^\dagger c_j^\dagger = \frac{1}{N}\sum_j\sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{R}_j\cdot\mathbf{k}}e^{-i\mathbf{R}_{j+\delta}\cdot\mathbf{k}'}c_{\mathbf{k}'}^\dagger c_{\mathbf{k}}^\dagger$$
$$= \sum_{\mathbf{k}} e^{i\mathbf{R}_\delta\cdot\mathbf{k}}c_{-\mathbf{k}}^\dagger c_{\mathbf{k}}^\dagger$$
$$= \sum_{\mathbf{k}} i\sin(\mathbf{R}_\delta\cdot\mathbf{k})c_{-\mathbf{k}}^\dagger c_{\mathbf{k}}^\dagger$$

so the momentums pace Dirac hamiltonian can be written as(in the Nambu spinor $\psi_k = (c_{\mathbf{k}}, c_{-\mathbf{k}}^\dagger)^T$)

$$H(\mathbf{k}) = (t\cos k_x + t\cos k_y + \frac{\mu}{2})\sigma_z + 2\Delta\sin k_x\sigma_x + 2\Delta\sin k_y\sigma_y$$

in this case, the product of the real Dirac matrices is $B = \sigma_x\sigma_z = -i\sigma_y$,and we choose $\gamma_0 = \sigma_z$, thus $A = B\gamma_0 = \sigma_x$ which serve as the particle hole symmetry which squares to $+1$. we can verify this explicitly:

$$\sigma_x H(-k)^T\sigma_x = -H(k)$$

similarly, in this case, we have n=1 and W=I and

$$2^n i^n \to \mathrm{tr}[\sigma_z\sigma_x\sigma_y] = 2i$$

so the topological operator is

$$\hat{C}_{2D-D} = \frac{4\pi}{2^n i^n} W(QXPYQ - PXQYP) = -2\pi i(QXPYQ - PXQYP)$$

∮.3 2D-DIII-Spin-less helical p-wave SC

similarly, as for the symmetry class DIII in 2 dimension, since it belongs to the first descendant series, we can consider it as the reduction from the same class in 3D dimension, which is

$$\begin{aligned} H(k) &= \Delta \sin k_x s_x \otimes \sigma_z + \Delta \sin k_y s_y \otimes I + (-\Delta \sin k_z) s_x \otimes \sigma_x \\ &+ \{2t(\cos k_x + \cos k_y + \cos k_z) - \mu\} s_z \otimes I \end{aligned}$$

and the real space lattice hamiltonian

$$\begin{aligned} H(k) &= \Delta \sin k_x s_x \otimes \sigma_z + \Delta \sin k_y s_y \otimes I + (-\Delta \sin k_z) s_x \otimes \sigma_x \\ &+ \{2t(\cos k_x + \cos k_y + \cos k_z) - \mu\} s_z \otimes I \\ \rightarrow H &= \Delta \sum_j (ic^\dagger_{j+x,\uparrow} c^\dagger_{j,\uparrow} - ic_{j,\uparrow} c_{j+x,\uparrow} - ic^\dagger_{j+x,\downarrow} c^\dagger_{j,\downarrow} + ic_{j,\downarrow} c_{j+x,\downarrow}) \\ &+ \Delta \sum_j (c^\dagger_{j+y,\uparrow} c^\dagger_{j,\uparrow} + c_{j,\uparrow} c_{j+y,\uparrow} + c^\dagger_{j+y,\downarrow} c^\dagger_{j,\downarrow} + c_{j,\downarrow} c_{j+y,\downarrow}) \\ &- \Delta \sum_j (ic^\dagger_{j+z,\downarrow} c^\dagger_{j,\uparrow} + ic^\dagger_{j+z,\uparrow} c^\dagger_{j,\downarrow} - ic_{j,\uparrow} c_{j+z,\downarrow} - ic_{j,\downarrow} c_{j+z,\uparrow}) \\ &+ 2t \sum_{\delta,j,\sigma} (c^\dagger_{j+\delta,\sigma} c_{j,\sigma} + c^\dagger_{j,\sigma} c_{j+\delta,\sigma}) \\ &- \mu \sum_j (c^\dagger_{j,\uparrow} c_{j,\uparrow} - c_{j,\uparrow} c^\dagger_{j,\uparrow} + c^\dagger_{j,\downarrow} c_{j,\downarrow} - c_{j,\downarrow} c^\dagger_{j,\downarrow}) \end{aligned}$$

by setting the coefficients $d_4(k) = -\Delta \sin k_z = 0$, or setting the term containing k_z to be zero.

that is turning off(ignoring) all the the term related to the z spatial dimension, namely, $\cos k_z, \sin k_z, c_{j+z}$ etc. in this case $n = [\frac{D+1+1}{2}] = 2$ and the the omitted Dirac matrices are Γ_4, Γ_5,thus $W = \Gamma_4\Gamma_5 = (s_x \otimes \sigma_x)(s_x \otimes \sigma_y) = iI \otimes \sigma_z$ and

$$2^n i^n \rightarrow \text{tr}[\Gamma_4\Gamma_5\Gamma_3\Gamma_1\Gamma_2] = -2^2$$

so the topological operator is

$$\hat{C}_{2D-DIII} = \frac{4\pi}{2^n i^n} W(QXPYQ - PXQYP) = -\pi W(QXPYQ - PXQYP)$$

∮.4 2D-AII

as for the symmetry class AII in 2 dimension, we follow the general strategy discussed above. since it belongs to the second descendant. we can consider it as the reduction from the symmetry class AII in 3 dimension, where we have discussed before.

$$\begin{aligned} H(k) &= (M + 2M_1 + 4M_2 - 2M_1 \cos k_z - 2M_2 \cos k_x - 2M_2 \cos k_y)(I \otimes \tau_z) \\ &+ B_0 \sin k_z I \otimes \tau_y + A_0 \sin k_y \sigma_x \otimes \tau_x - A_0 \sin k_x \sigma_y \otimes \tau_x \end{aligned}$$

and the real space lattice version one

$$H = -\mu \sum_{j,\alpha,\sigma} c^\dagger_{j,\alpha,\sigma} c_{j,\alpha,\sigma} + (M + 2M_1 + 4M_2) \sum_j (c^\dagger_{j,s,\uparrow} c_{j,s,\uparrow} - c^\dagger_{j,p,\uparrow} c_{j,p,\uparrow} + c^\dagger_{j,s,\downarrow} c_{j,s,\downarrow} - c^\dagger_{j,p,\downarrow} c_{j,p,\downarrow})$$

$$- \frac{A_0}{2} \sum_j \{-(c^\dagger_{j,s,\uparrow} c_{j+x,p,\downarrow} - c^\dagger_{j+x,s,\uparrow} c_{j,p,\downarrow}) - (c^\dagger_{j,p,\uparrow} c_{j+x,s,\downarrow} - c^\dagger_{j+x,p,\uparrow} c_{j,s,\downarrow})\} + H.c$$
$$- i\frac{A_0}{2} \sum_j \{(c^\dagger_{j,s,\uparrow} c_{j+y,p,\downarrow} - c^\dagger_{j+y,s,\uparrow} c_{j,p,\downarrow}) + (c^\dagger_{j,p,\uparrow} c_{j+y,s,\downarrow} - c^\dagger_{j+y,p,\uparrow} c_{j,s,\downarrow})\} + H.c$$
$$+ \frac{B_0}{2} \sum_j \{-(c^\dagger_{j,s,\uparrow} c_{j+z,p,\uparrow} - c^\dagger_{j+z,s,\uparrow} c_{j,p,\uparrow}) - (c^\dagger_{j,s,\downarrow} c_{j+z,p,\downarrow} - c^\dagger_{j+z,s,\downarrow} c_{j,p,\downarrow})\} + H.c$$
$$- M_1 \sum_{j,\delta=z} \{c^\dagger_{j,s,\uparrow} c_{j+\delta,s,\uparrow} - c^\dagger_{j,p,\uparrow} c_{j+\delta,p,\uparrow} + c^\dagger_{j,s,\downarrow} c_{j+\delta,s,\downarrow} - c^\dagger_{j,p,\downarrow} c_{j+\delta,p,\downarrow}\} + H.c$$
$$- M_2 \sum_{j,\delta=x,y} \{c^\dagger_{j,s,\uparrow} c_{j+\delta,s,\uparrow} - c^\dagger_{j,p,\uparrow} c_{j+\delta,p,\uparrow} + c^\dagger_{j,s,\downarrow} c_{j+\delta,s,\downarrow} - c^\dagger_{j,p,\downarrow} c_{j+\delta,p,\downarrow}\} + H.c$$

by setting the $d_4(k) = B_0 \sin k_z = 0$, namely by choosing $B_0 = 0$.

in this case n=2 and $W = \Gamma_3\Gamma_4 = i\sigma_z \otimes \tau_z$ and

$$2^n i^n \to \mathrm{tr}[\Gamma_3\Gamma_4\Gamma_5\Gamma_2\Gamma_1] = 2^2$$

so the topological operator is

$$\hat{C}_{2D-AII} = \frac{4\pi}{2^n i^n} W(QXPYQ - PXQYP) = \pi\sigma_z \otimes \tau_z(QXPYQ - PXQYP)$$

∮.5 2D-C

as for symmetry class C in 2D, since it belongs to the even series, we should consider the Clifford algebra Cl^5. and pick up 3 out of five to serve as the basis for the Dirac hamiltonian. since only three are need and we know the particle-hole operator behaves like $i\sigma_y K$(in algebra Cl^5, the product of all the real matrices is $-i\sigma_y \otimes \tau_z$). thus we can regrard $\sigma_x, \sigma_y, \sigma_z$ as the chosen three and consider the particle-hole operator as $i\sigma_y K$ and thus then try to construct the Dirac hamiltonian.

$$H(k) = d_1\sigma_x + d_2\sigma_y + d_3\sigma_z$$

in this case, we don not following the strategy used in the general discussion, so the constrain in the coefficient $d_i(k)$ is not the same as before(where we use Cl^5 to describe the even series).

in the following, we try to consider the constrain that with these coefficients, at first

$$\begin{aligned}
\sigma_y K(H(-k))\sigma_y K &= \sigma_y H(-k)^*\sigma_y = \sigma_y H(-k)^T\sigma_y \\
&= \sigma_y(d_1(-k)\sigma_x - d_2(-k)\sigma_y + d_3(-k)\sigma_z)\sigma_y \\
&= -(d_1(-k)\sigma_x + d_2(-k)\sigma_y + d_3(-k)\sigma_z) \\
&= -H(k)
\end{aligned}$$

so the particle-hole symmetry constrain would require that all the coefficients $d_i(k)$ should be even on k. so we can write down the hamiltonian in the momentum space as[15]

$$H(k) = (\frac{\mu}{2} + 2t\cos k_x + 2t\cos k_y)\sigma_z + \Delta(2\cos k_x - \cos k_y)\sigma_x + 2\Delta \sin k_x \sin k_y \sigma_y$$

where we have also used the Nambu spinor $\psi_k = (c_{\mathbf{k}}, c^\dagger_{-\mathbf{k}})^T$, in order to convert it to the real space lattice hamiltonian, we have

$$(\frac{\mu}{2} + 2t\cos k_x + 2t\cos k_y)\sigma_z = \sum_{j,\delta} t(c^\dagger_j c_{j+\delta} + c^\dagger_{j+\delta} c_j) - \mu \sum_j c^\dagger_j c_j$$

where we have already done for the class D in 2D.

using the fact that has been derived before

$$\begin{aligned}\sum_j c_j c_{j+\delta} &= \frac{1}{N}\sum_j \sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{R}_j\cdot\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{R}_{j+\delta}\cdot\mathbf{k}'} c_{\mathbf{k}'} \\ &= \sum_{\mathbf{k}} e^{-i\mathbf{R}_\delta\cdot\mathbf{k}} c_{\mathbf{k}} c_{-\mathbf{k}} \\ &= \sum_{\mathbf{k}} -i\sin(\mathbf{R}_\delta\cdot\mathbf{k}) c_{\mathbf{k}} c_{-\mathbf{k}}\end{aligned}$$

$$\begin{aligned}\sum_j c^\dagger_{j+\delta} c^\dagger_j &= \frac{1}{N}\sum_j \sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{R}_j\cdot\mathbf{k}} e^{-i\mathbf{R}_{j+\delta}\cdot\mathbf{k}'} c^\dagger_{\mathbf{k}'} c^\dagger_{\mathbf{k}} \\ &= \sum_{\mathbf{k}} e^{i\mathbf{R}_\delta\cdot\mathbf{k}} c^\dagger_{-\mathbf{k}} c^\dagger_{\mathbf{k}} \\ &= \sum_{\mathbf{k}} i\sin(\mathbf{R}_\delta\cdot\mathbf{k}) c^\dagger_{-\mathbf{k}} c^\dagger_{\mathbf{k}}\end{aligned}$$

we can find that

$$\begin{aligned}(2\cos k_x - \cos k_y)\sigma_x &= \sum_j (c_j c_{j+x} + c_j c_{j-x} + c^\dagger_{j+x} c^\dagger_j + c^\dagger_{j-x} c^\dagger_j) \\ &+ \sum_j (-c_j c_{j+x} - c_j c_{j-x} - c^\dagger_{j+x} c^\dagger_j - c^\dagger_{j-x} c^\dagger_j)\end{aligned}$$

as for the term $\sin k_x \sin k_y$, we should notice that $\sin k_x \sin k_y = \frac{1}{2i}(e^{ik_x} - e^{-ik_x})\frac{1}{2i}(e^{ik_y} - e^{-ik_y}) = -\frac{1}{4}(e^{i(k_x+k_y)} - e^{i(k_x-k_y)} - e^{i(-k_x+k_y)} + e^{i(-k_x-k_y)})$ thus we have(using the fact $\sum_j c^\dagger_j c^\dagger_{j+\delta} = \sum_{\mathbf{k}} e^{i\mathbf{R}_\delta\cdot\mathbf{k}} c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}$)

$$\begin{aligned}2\sin k_x \sin k_y \sigma_y &= -2i\sin k_x \sin k_y c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} + H.c \\ &= \frac{i}{2}(e^{i(k_x+k_y)} - e^{i(k_x-k_y)} - e^{i(-k_x+k_y)} + e^{i(-k_x-k_y)}) c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} + H.c \\ &= \frac{i}{2}(c^\dagger_j c^\dagger_{j+x+y} + c^\dagger_j c^\dagger_{j+x-y} - c^\dagger_j c^\dagger_{j-x+y} + c^\dagger_j c^\dagger_{j-x-y}) + H.c\end{aligned}$$

collecting all the term, the real space lattice hamiltonian can be written as [15]

$$\begin{aligned}H &= \sum_{j,\delta} t(c^\dagger_j c_{j+\delta} + c^\dagger_{j+\delta} c_j) - \mu \sum_j c^\dagger_j c_j \\ &+ \Delta \sum_j (c_j c_{j+x} + c_j c_{j-x} + c^\dagger_{j+x} c^\dagger_j + c^\dagger_{j-x} c^\dagger_j) \\ &+ \Delta \sum_j (-c_j c_{j+x} - c_j c_{j-x} - c^\dagger_{j+x} c^\dagger_j - c^\dagger_{j-x} c^\dagger_j) \\ &+ \{\frac{i}{2}\Delta(c^\dagger_j c^\dagger_{j+x+y} + c^\dagger_j c^\dagger_{j+x-y} - c^\dagger_j c^\dagger_{j-x+y} + c^\dagger_j c^\dagger_{j-x-y}) + H.c\}\end{aligned}$$

in this case, we have n=1 and W=I, so the topological operator is

$$\hat{C}_{2D-C} = \frac{4\pi}{2^n i^n} W(QXPYQ - PXQYP) = -2\pi i(QXPYQ - PXQYP)$$

ℜ.4 Explicit form for different symmetry classes in one dimension

∮.1 1D-AIII

as for the 1D AIII class, the Clifford algebra is Cl^3, the Dirac matrices are just the three Pauli matrices, and the Dirac hamiltonian in momentum space can be written as

$$H(k) = Ak_x\sigma_x + (M + Bk_x^2)\sigma_z = A\sin k_x\sigma_x + (M + 2B - 2B\cos k_x)\sigma_z$$

the chiral operator is chosen to be the $S = \sigma_y$.and we can convert the above one to the real space lattice hamiltonian

$$
\begin{aligned}
H = &\sum_j (M + 2B)(c^\dagger_{j,1}c_{j,1} - c^\dagger_{j,2}c_{j,2}) \\
&\sum_j -B(c^\dagger_{j,1}c_{j+x,1} + c^\dagger_{j,1}c_{j-x,1} - c^\dagger_{j,2}c_{j+x,2} - c^\dagger_{j,2}c_{j-x,2}) \\
&+ \frac{A}{2}\sum_j -i(c^\dagger_{j,1}c_{j+x,2} - c^\dagger_{j,1}c_{j-x,2}) - i(c^\dagger_{j,2}c_{j+x,1} - c_{j,2})^\dagger c_{j-x,1}
\end{aligned}
$$

in this class,$W = S = \sigma_y$ and n=1 and

$$2^n i^n \rightarrow \mathrm{tr}[\sigma_y\sigma_z\sigma_x] = 2i$$

so the topological operator is

$$\hat{C}_{1D-AIII} = i\frac{2}{2^n i^n}W(QXP + PXQ) = +1\sigma_y(QXP + PXQ)$$

∮.2 1D-BDI-spinless SuSchrieffer-Heeger model

the example of symmetry class BDI in 1D is the spin-less SSH model, with 2 lattice site A and B in each unite cell, A is the label for the creation of electron-like quasi-particle, B is the label for the creation of hole-like quasi-particle, and the lattice hamiltonian can be written as[16]

$$H = \sum_j (t + \delta t)c^\dagger_{j,A}c_{j,B} + (t - \delta t)c^\dagger_{j+x,A}c_{j,B} + H.c$$

in the spinor $\psi_k = (c_{k,A}, c_k, B)$, the momentum space hamiltonian can be written as

$$
\begin{aligned}
H(k) &= (t + \delta t)\sigma_x + (t - \delta t)\cos k_x\sigma_x + (t - \delta t)\sin k_x\sigma_y \\
&= (t + \delta t + (t - \delta t)\cos k_x)\sigma_x + (t - \delta t)\sin k_x\sigma_y
\end{aligned}
$$

in this case, the product of all the real matrices is $B = \sigma_z\sigma_x = i\sigma_y$ and the chiral operator is chosen to be $S = \sigma_z$, $\gamma_0 = \sigma_x$, thus $A = B\gamma_0 = \sigma_z$, so the the particle-hole operator is $P = \sigma_z K$, and the time reversal operator is $T = IK$, which both square to +1.

in this class n=1 and $W = \sigma_z$ and

$$2^n i^n \rightarrow \mathrm{tr}[\sigma_z\sigma_x\sigma_y] = 2i$$

so the topological operator is

$$\hat{C}_{1D-BDI} = i\frac{2}{2^n i^n}W(QXP + PXQ) = +1\sigma_z(QXP + PXQ)$$

∮.3 1D-D-spinless Kitaev p-wave SC chain

as for the 1D class D, we can derive it from 2D class D by induction(setting $k_y = 0$), or we can consider another concrete example, that is the spin-less Kitaev p-wave SC chain described by[17]

$$
\begin{aligned}
H = &\sum_j t(c^\dagger_j c_{j+x} + c^\dagger_{j+x}c_j) - \mu\sum_j c^\dagger_j c_j \\
&+ \sum_j \Delta(c_j c_{j+x} + c^\dagger_{j+x}c^\dagger_j)
\end{aligned}
$$

we can use the Nambu spinor $\psi_j = (c_j, c_j^\dagger)^T$ or $\psi_k = (c_k, c_{-k}^\dagger)^T$, then convert this real space lattice hamiltonian to the momentum space version

$$H(k) = (t\cos k - \mu)\sigma_z + \Delta \sin k \sigma_y$$

the particle hole operator is given by $P = \sigma_x K$, we can check that

$$PH(-k)P = -H(k)$$

in this case n=1 and $W = \sigma_x$ and

$$2^n i^n \to \mathrm{tr}[\sigma_x\sigma_z\sigma_y] = -2i$$

so the topological operator is

$$\hat{C}_{1D-D} = i\frac{2}{2^n i^n} W(QXP + PXQ) = -1\sigma_x(QXP + PXQ)$$

∮.4 1D-DIII

as for the symmetry class DIII in 1D, since it's in the Second Descendant, it can be viewed as the reduction from the same class in 2D, that is setting the term involve k_y to vanishing in the Dirac model for 2D DIII(which is also a reduction from the 3D DIII class.)

$$\begin{aligned}H(k) &= \Delta \sin k_x s_x \otimes \sigma_z + \Delta \sin k_y s_y \otimes I + (-\Delta \sin k_z) s_x \otimes \sigma_x \\ &+ \{2t(\cos k_x + \cos k_y + \cos k_z) - \mu\} s_z \otimes I\end{aligned}$$

and the real space lattice hamiltonian

$$\begin{aligned}H(k) &= \Delta \sin k_x s_x \otimes \sigma_z + \Delta \sin k_y s_y \otimes I + (-\Delta \sin k_z) s_x \otimes \sigma_x \\ &+ \{2t(\cos k_x + \cos k_y + \cos k_z) - \mu\} s_z \otimes I \\ \rightarrow H &= \Delta \sum_j (ic_{j+x,\uparrow}^\dagger c_{j,\uparrow}^\dagger - ic_{j,\uparrow}c_{j+x,\uparrow} - ic_{j+x,\downarrow}^\dagger c_{j,\downarrow}^\dagger + ic_{j,\downarrow}c_{j+x,\downarrow}) \\ &+ \Delta \sum_j (c_{j+y,\uparrow}^\dagger c_{j,\uparrow}^\dagger + c_{j,\uparrow}c_{j+y,\uparrow} + c_{j+y,\downarrow}^\dagger c_{j,\downarrow}^\dagger + c_{j,\downarrow}c_{j+y,\downarrow}) \\ &- \Delta \sum_j (ic_{j+z,\downarrow}^\dagger c_{j,\uparrow}^\dagger + ic_{j+z,\uparrow}^\dagger c_{j,\downarrow}^\dagger - ic_{j,\uparrow}c_{j+z,\downarrow} - ic_{j,\downarrow}c_{j+z,\uparrow}) \\ &+ 2t \sum_{\delta,j,\sigma} (c_{j+\delta,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+\delta,\sigma}) \\ &- \mu \sum_j (c_{j,\uparrow}^\dagger c_{j,\uparrow} - c_{j,\uparrow}c_{j,\uparrow}^\dagger + c_{j,\downarrow}^\dagger c_{j,\downarrow} - c_{j,\downarrow}c_{j,\downarrow}^\dagger)\end{aligned}$$

by setting the coefficients involving k_y, k_z to be zero in the momentum space hamiltonian or any term involving y and z in the real space lattice hamiltonian.

in this case, we have n=2 and $W = (s_y \otimes I)(s_x \otimes \sigma_x)(s_x \otimes \sigma_y) = is_y \otimes \sigma_z$ and

$$2^n i^n \to \mathrm{tr}[\Gamma_2\Gamma_4\Gamma_5\Gamma_3\Gamma_1] = -2^2$$

so the topological operator is

$$\hat{C}_{1D-DIII} = i\frac{2}{2^n i^n} W(QXP + PXQ) = -\frac{i}{2} is_y \otimes \sigma_z(QXP + PXQ)$$

∮.5 1D-CII

as for the symmetry class CII in 1D, since it belongs to the even series, if we follow the standard strategy, then we should consider the Clifford algebra Cl^5, the Dirac matrices are given by

$$\Gamma_{1\sim 5} = \{\sigma_x \otimes \tau_z, \sigma_y \otimes \tau_z, I \otimes \tau_x, I \otimes \tau_y, \sigma_z \otimes \tau_z\}$$

the product of all the real matrices is $B = i\sigma_y \otimes \tau_x$, if we chose the chiral operator as $S = \Gamma_5 = \sigma_z \otimes \tau_z$, then the $BS = \sigma_x \otimes \tau_y$. so the time reversal operator is $T = \sigma_x \otimes \tau_y K$ and the particle-hole operator is $P = i\sigma_y \otimes \tau_x$ which both square to -1.

if we choose $\gamma_0 = -i(\sigma_x \otimes \tau_z)(I \otimes \tau_x)(I \otimes \tau_y) = \sigma_x \otimes I$, then the Dirac hamiltonian can be written as

$$H(k) = d_1(k)\sigma_y \otimes \tau_z + d_0(k)\sigma_x \otimes I$$

this is the standard procedure. But we can choose new basis, so that our $T = i\sigma_y K$ and $P = i\tau_y K$, and then we can find the following form hamiltonian take this two symmetries:

$$H(k) = d_1(k)\sigma_y \otimes \tau_z + d_0(k) I \otimes \tau_x$$

we can set $d_1 = A\sin k$, $d_2 = M + 2B - 2B\cos k(d_1 \sim k, d_2 \sim M + k^2$ in the linear expansion) then using the spinor $\psi_k = (c_{k,1}, c_{k,2}, c_{k,3}, c_{k,4})^T$, we can convert this momentum space hamiltonian to the real space one with the same process discussed above

$$\begin{aligned}
H(k) &= A\sin k(-ic^\dagger_{k,1}c_{k,3} + ic^\dagger_{k,2}c_{k,4} + ic^\dagger_{k,3}c_{k,1} - ic^\dagger_{k,4}c_{k,2}) \\
&\quad + (M + 2B - 2B\cos k)(c^\dagger_{k,1}c_{k,2} + c^\dagger_{k,2}c_{k,1} + c^\dagger_{k,3}c_{k,4} + c^\dagger_{k,4}c_{k,3}) \\
&\to \frac{A}{2}\sum_j\{-(c^\dagger_{j,1}c_{j+x,3} - c^\dagger_{j,1}c_{j-x,3}) + (c^\dagger_{j,2}c_{j+x,4} - c^\dagger_{j,2}c_{j-x,4}) + (c^\dagger_{j,3}c_{j+x,1} - c^\dagger_{j,3}c_{j-x,1}) - (c^\dagger_{j,4}c_{j+x,2} - c^\dagger_{j,4}c_{j-x,2})\} \\
&(M + 2B)\sum_j(c^\dagger_{j,1}c_{j,2} + c^\dagger_{j,2}c_{j,1} + c^\dagger_{j,3}c_{j,4} + c^\dagger_{j,4}c_{j,3}) \\
&\quad - B\sum_j\{(c^\dagger_{j,1}c_{j+x,2} + c^\dagger_{j,1}c_{j-x,2}) + (c^\dagger_{j,2}c_{j+x,1} + c^\dagger_{j,2}c_{j-x,1}) + (c^\dagger_{j,3}c_{j+x,4} + c^\dagger_{j,3}c_{j-x,4}) + (c^\dagger_{j,4}c_{j+x,3} + c^\dagger_{j,4}c_{j-x,3})\}
\end{aligned}$$

in this case, n=2, and $W = \Gamma_1\Gamma_4\Gamma_5 = i\sigma_y \otimes \tau_y = -iS$ and

$$2^n i^n \to \text{tr}[\Gamma_1\Gamma_4\Gamma_5\Gamma_3\Gamma_2] = 2^2$$

so the topological operator is

$$\hat{C}_{1D-CII} = i\frac{2}{2^n i^n}W(QXP + PXQ) = i\frac{1}{2}W(QXP + PXQ)$$

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