Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

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A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.

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The quantum Hall effect[1] (QHE) in two-dimensional (2D) electron systems is usually associated with the presence of a uniform externally generated magnetic field, which splits the spectrum of electron energy levels into Landau levels. In this Letter I show how, in principle, a QHE may also result from breaking of time-reversal symmetry (i.e., magnetic ordering) *without* any net magnetic flux through the unit cell of a periodic 2D system. In this case, the electron states retain their usual Bloch state character.

The model presented here is also interesting in that if its parameters are on a critical line at which its ground state changes from the normal semiconductor state to this new type of QHE state, its low-energy states simulate a "(2+1)-dimensional" relativistic quantum field theory exhibiting the so-called "parity anomaly"[2] and a (2+1)-D analog of "chiral" fermions *without* the opposite-chirality anomaly-canceling partners[3] that usually accompany them in lattice realizations of field theories ("fermion doubling").

In the zero-temperature limit, the transverse conductivity σ^{xy} of a periodic 2D electron system with a gap in the single-particle density of states at the Fermi level takes quantized values $\nu e^2/h$, where ν is generally rational, but can only take *integer* values in the absence of electron interactions.[4] This property of a pure system is stable against sufficiently weak disorder effects. Since σ^{xy} is odd under time reversal, a nonzero value can only occur if time-reversal invariance is broken.

In the usual QHE, the gap at the Fermi level results from the splitting of the spectrum into Landau levels by an external magnetic field. The scenario considered here is different, and involves a 2D semimetal where there is a degeneracy at isolated points in the Brillouin zone between the top of the valence band and the bottom of the conduction band, that is associated with the presence of both inversion symmetry and time-reversal invariance. If inversion symmetry is broken, a gap opens and the system becomes a normal semiconductor ($\nu=0$), but if the gap opens because time-reversal invariance is broken the system becomes a $\nu=\pm 1$ integer QHE state. If both perturbations are present, their relative strengths determine which type of state is realized.

To model a 2D semimetal, I use the "2D graphite" model investigated previously by Semenoff[5] as a possible lattice realization of a (2+1)-D field theory with the anomaly. 2D graphite has the honeycomb net structure, consisting of two interpenetrating triangular lattices ("A" and "B" sublattices) with one lattice point of each type per unit cell (Fig. 1). A 2D inversion (i.e., a rotation in the plane by π) interchanges the two sublattices. Since spin-orbit coupling effects will not be included, the electron spin will (for the moment) be suppressed.

Semenoff[5] investigated the tight-binding model with one orbital per site and a real hopping matrix element t_1 between nearest neighbors on different sublattices, and also considered the effect of an inversion-symmetry-breaking on-site energy $+M$ on A sites and $-M$ on B sites. The model has point group C_{6v} ($M=0$) or C_{3v} ($M\neq 0$). In this original version of the model, time-reversal invariance is present, and Semenoff[5] found complete cancellation of the anomaly in the $M=0$ model due to fermion doubling, and normal semiconductor behavior for $M\neq 0$.

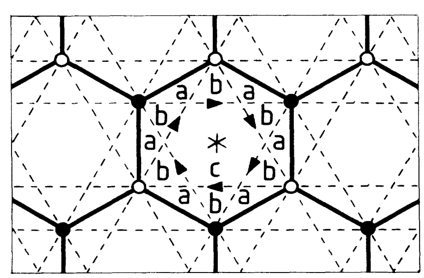

FIG. 1. The honeycomb-net model ("2D graphite") showing nearest-neighbor bonds (solid lines) and second-neighbor bonds (dashed lines). Open and solid points, respectively, mark the A and B sublattice sites. The Wigner-Seitz unit cell is conveniently centered on the point of sixfold rotation symmetry (marked "*") and is then bounded by the hexagon of nearest-neighbor bonds. Arrows on second-neighbor bonds mark the directions of positive phase hopping in the state with broken time-reversal invariance.

I now include a second real hopping term t_2 between second-neighbor sites (i.e., between nearest-neighbor sites on the *same* sublattice). This does not change the space group, though it does eliminate a particle-hole symmetry of the energy bands of the original model. To break time-reversal invariance, I also add a periodic local magnetic-flux density $B(\mathbf{r})$ in the $\hat{\mathbf{z}}$ direction normal to the 2D plane, with the full symmetry of the lattice, and with *zero total flux through the unit cell.*

Since the net flux per unit cell vanishes, the vector potential $\mathbf{A}(\mathbf{r})$ can be chosen to be periodic. The effect of this local field is to multiply the matrix element for hopping between sites by the unimodular phase factor $\exp[i(e/\hbar)\int \mathbf{A}\cdot d\mathbf{r}]$ where the integral is along the hopping path, which I take to be rectilinear. The phases can be chosen with any consistent convention such that the total phase accumulated around a closed path adds up to the flux enclosed in units of the flux quantum $\Phi_0 = |h/e|$.

Since closed paths of first-neighbor hops enclose complete unit cells (and hence no net flux) the t_1 matrix elements are unaffected. The t_2 matrix elements acquire a phase $\phi = 2\pi(2\Phi_a + \Phi_b)/\Phi_0$, where Φ_a and Φ_b are the fluxes through the regions of the unit cell marked a and b in Fig. 1. The hopping directions for which the amplitudes are $t_2\exp(+i\phi)$ are shown in Fig. 1: It can be seen that the Hamiltonian has acquired a chirality if the local field is present.

It is useful to consider a possible model for the origin of such an internal magnetic field. Magnetic dipole moments μ, ordered ferromagnetically normal to the plane, are placed at the center of each hexagonal cell of the honeycomb net, and $B(\mathbf{r})$ is the sum of their dipole fields. Note that ferromagnetic ordering in 2D does *not* generate a uniform component of the magnetic-flux density. The absolute value of ϕ is $C\alpha^2\mu/a$, where α is the fine-structure constant, μ is the dipole moment in Bohr magneton units, a is the lattice spacing Bohr radii, and C is a dimensionless constant (of order unity) which depends on the lattice structure. If μ and a are of order unity in their natural units, the phase ϕ in this model will be a small quantity controlled by the fine-structure constant.

To diagonalize the Hamiltonian, I use a basis of two-component "spinors" $(\psi_{\mathbf{k}A},\psi_{\mathbf{k}B})$ of Bloch states constructed on the two sublattices. Let $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$ be the displacements from a B site to its three nearest-neighbor A sites, defined so that $\hat{\mathbf{z}}\cdot\mathbf{a}_1\times\mathbf{a}_2$ is positive. I also define $\mathbf{b}_1=\mathbf{a}_2-\mathbf{a}_3$, $\mathbf{b}_2=\mathbf{a}_3-\mathbf{a}_1$, etc.; the set of displacements to the six nearest neighbors on the same sublattice is $\{\pm\mathbf{b}_i\}$. In this representation, the Hamiltonian becomes

$$\mathbf{H}(\mathbf{k}) = 2t_2\cos\phi\left(\sum_i \cos(\mathbf{k}\cdot\mathbf{b}_i)\right)\mathbf{I} + t_1\left(\sum_i [\cos(\mathbf{k}\cdot\mathbf{a}_i)\boldsymbol{\sigma}^1 + \sin(\mathbf{k}\cdot\mathbf{a}_i)\boldsymbol{\sigma}^2]\right) + \left[M - 2t_2\sin\phi\left(\sum_i \sin(\mathbf{k}\cdot\mathbf{b}_i)\right)\right]\boldsymbol{\sigma}^3, \quad (1)$$

where $\boldsymbol{\sigma}^i$ are Pauli matrices. The Brillouin zone is a hexagon rotated 90° with respect to the Wigner-Seitz unit cell: At its six corners $(\mathbf{k}\cdot\mathbf{a}_1, \mathbf{k}\cdot\mathbf{a}_2, \mathbf{k}\cdot\mathbf{a}_3)$ is a permutation of $(0, 2\pi/3, -2\pi/3)$. The two distinct corners $\mathbf{k}^0_\alpha$ are defined so that $\mathbf{k}^0_\alpha\cdot\mathbf{b}_i = (2\pi/3)\alpha$, $\alpha=\pm 1$.

The energy bands are easily obtained. There are two bands which only touch if all three Pauli matrix terms in (1) have vanishing coefficients. This can only occur at zone corners $\mathbf{k}^0_\alpha$, and then only if $M = 3\sqrt{3}\alpha t_2\sin\phi$. I will assume $|t_2/t_1| < \frac{1}{3}$, which guarantees that the two bands never overlap, and are separated by a finite gap unless they touch.

If both M and $t_2\sin\phi$ vanish, the bands touch at both zone corners, where the group of the wave vector[6] has the unitary subgroup C_{3v}, which contains a reflection that interchanges the A and B sublattices. Apart from the zone center, these are the only points in the Brillouin zone where this group has irreducible representations with dimensions greater than unity, and the degenerate states at these points belong to the two-dimensional representation. The touching of the bands at *two* distinct points in the Brillouin zone is a manifestation of fermion doubling.[3,5] The degeneracy of the bands at these points is lifted either by nonzero M or nonzero $t_2\sin\phi$, either of which reduce the unitary subgroup to C_3, which has only one-dimensional irreducible representations.

When the Fermi level lies in a gap between two bands, σ_{xy} is quantized at $T=0$, and its value can be obtained through the thermodynamic relation[7] $\sigma^{xy} = \partial\sigma/\partial B_0|_{\mu,T}$ evaluated at $B_0 = 0$, where σ is the 2D electric-charge density, and B_0 is the flux density of a uniform external magnetic field in the $\hat{\mathbf{z}}$ direction. To calculate the induced charge density σ to a weak external magnetic field, it is convenient to expand the Hamiltonian in the neighborhood of the band extrema at the zone corners $\mathbf{k}^0_\alpha$ to linear order in $\delta\mathbf{k} = \mathbf{k} - \mathbf{k}^0_\alpha$, and make the Landau-Peierls substitution $\hbar\delta\mathbf{k}\rightarrow\boldsymbol{\Pi}$, where $\boldsymbol{\Pi} = (\Pi^x, \Pi^y)$ is the dynamical momentum with components satisfying the commutation relation $[\Pi^x, \Pi^y] = i\hbar e B_0$.

For weak B_0, coupling between the two distinct zone corners can be neglected, and two independent effective Hamiltonians $\mathbf{H}_\alpha$ are obtained, where

$$\mathbf{H}_\alpha = c(\Pi^1_\alpha\boldsymbol{\sigma}^2 - \Pi^2_\alpha\boldsymbol{\sigma}^1) + m_\alpha c^2\boldsymbol{\sigma}^3. \quad (2)$$

Here $c = \frac{3}{2}t_1|\mathbf{a}_i|/\hbar$ and $m_\alpha c^2 = M - 3\sqrt{3}\alpha t_2\sin\phi$; Π^1_α and Π^2_α are Hermitian operators with the commutation relation $[\Pi^1_\alpha, \Pi^2_\alpha] = i\alpha e B_0\hbar$, defined by

$$(\Pi^1_\alpha + i\Pi^2_\alpha) = \tfrac{2}{3}\sum_i e^{-i\mathbf{k}^0_\alpha\cdot\mathbf{a}_i}(\mathbf{a}_i\cdot\boldsymbol{\Pi})/|a_i|. \quad (3)$$

After second-quantization, (2) is precisely the Hamiltonian of a free-fermion-field theory studied by Jackiw[2] as an (2+1)-D analog of the Dirac Hamiltonian.

The spectrum of (2) is relativistic; for $B_0 = 0$,

$$\epsilon_{\alpha\pm}(\mathbf{k}) = \pm[(\hbar ck)^2 + (m_\alpha c^2)^2]^{1/2},$$

while for $B_0 \neq 0$, relativistic Landau levels are obtained[2] as follows:

$$\epsilon_{an\pm} = \pm[(m_a c^2)^2 + n\hbar |eB_0| c^2]^{1/2} \quad (n \geq 1), \tag{4a}$$

$$\epsilon_{a0} = a m_a c^2 \operatorname{sgn}(eB_0). \tag{4b}$$

Every $n \geq 1$ level that evolves out of the upper band as B_0 is turned on is balanced by a level that evolves from the lower band. However, the $n=0$ "zero-mode" energy is not symmetric under $B_0 \to -B_0$: It evolves from the *upper* band if $am_a eB_0$ is *positive,* and from the *lower* band if it is *negative.*

In the time-reversal symmetric case $t_2 \sin\phi = 0$, the two masses m_+ and m_- are equal, and the *sum* of the Landau-level spectra derived from the two distinct zone corners is particle-hole symmetric, *and* invariant under $B_0 \to -B_0$. In this case, $\sigma^{xy}=0$ by time-reversal invariance. As the Hamiltonian is changed, σ^{xy} remains invariant, provided the Fermi level remains in a gap.[4] When $B_0=0$, models where the Fermi level is in the gap and m_+ and m_- have the same sign can evolve continuously from the time-reversal invariant case, and hence have $\sigma^{xy}=0$.

To calculate σ^{xy} for models where m_+ and m_- have *opposite* signs, I continuously turn on the external field, then vary m_+ and m_- until they become equal, at the same time varying the Fermi level so at all times it lies in a gap. Comparison of the occupation numbers of the Landau levels obtained this way with those obtained by continuously applying the field to the time-reversal invariant system shows that they differ by the complete filling of one Landau level. Thus at $T=0$ and with a fixed chemical potential, the application of a weak external magnetic field to a system where m_+ and m_- have opposite signs induces an extra *field-dependent* ground-state charge density $\Delta\sigma = \pm e^2 B_0/h$ relative to the field-independent charge density when these parameters have the same sign. This allows σ^{xy} in the limit $B_0=0$ to be evaluated as $\nu e^2/h$, where $\nu = \frac{1}{2}[\operatorname{sgn}(m_-) - \operatorname{sgn}(m_+)] = \pm 1$ or 0. The phase diagram of ν for the spinless electron model as a function of M/t_2 and ϕ is shown in Fig. 2.

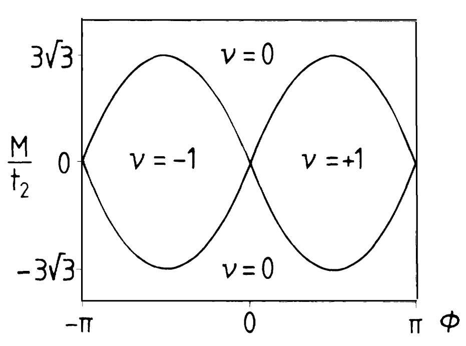

FIG. 2. Phase diagram of the spinless electron model with $|t_2/t_1| < \frac{1}{3}$. Zero-field quantum Hall effect phases ($\nu = \pm 1$, where $\sigma^{xy} = \nu e^2/h$) occur if $|M/t_2| < 3\sqrt{3}|\sin\phi|$. This figure assumes that t_2 is positive; if it is negative, ν changes sign. At the phase boundaries separating the anomalous and normal ($\nu=0$) semiconductor phases, the low-energy excitations of the model simulate undoubled massless chiral relativistic fermions.

I note that when the model has *neither* an inversion center *nor* time-reversal invariance (i.e., when both M and $t_2\sin\phi$ are nonzero), so $|m_+| \neq |m_-|$, the spectrum is no longer invariant under $\mathbf{k} \to -\mathbf{k}$, and the fermion-doubling principle is defeated. In particular, along the critical lines in the phase diagram where one of m_+ or m_- vanishes, the model has a low-lying massless spectrum simulating nondegenerate relativistic *chiral* fermions.

When $m_a=0$, the fermion field theory derived from the expansion (2) about the Fermi point with vanishing gap has a charge-conjugation symmetry (particle-hole symmetry) which is *not* present in the lattice model with $t_2 \neq 0$ from which it is derived. In the continuum field theory, there is no lower bound to the Dirac sea of filled electron states, and the establishment of *absolute* as opposed to *relative* values of σ^{xy} is ambiguous. Jackiw[2] invokes the charge-conjugation symmetry of (2) with $m_a=0$ to assign the value $\sigma^{xy}=0$ in the case of a particle-hole symmetric Fermi level, where the "zero-mode" Landau level (4b) is half filled. This would imply a quantum Hall effect with $\nu = \frac{1}{2}\alpha$ if the zero mode is filled, and $\nu = -\frac{1}{2}\alpha$ if it is empty. This suggests "charge fractionalization," and violates the principle that a noninteracting electron system can only exhibit an *integral* QHE. The model studied here shows how the high-energy cutoff structure of a model with undoubled fermions described by the relativistic Hamiltonian (2) at low energies *must* break the charge-conjugation symmetry, and give an extra contribution of $\pm\frac{1}{2}$ to ν, restoring an integral QHE. Thus even if the low-energy spectrum consists of undoubled chiral fermions, their partners must be present at high energies to restore a properly integral QHE.

When electron spin is included without any other change, there is an equal contribution from both spin components, and σ^{xy} is doubled. However, a periodic local magnetic field with the full symmetry of the lattice will also couple to electrons with a Zeeman term $H' = \gamma\phi S^z$, where S^z is the azimuthal electron spin. This term will relatively displace the up-spin and down-spin bands by an energy $\gamma\hbar\phi$, and if this exceeds the gap at the Fermi level, the system will become a partially spin-polarized metal. If $\frac{1}{2}|\gamma|\hbar$ exceeds $3\sqrt{3}|t_2|$, the QHE phases are completely eliminated, but if it is smaller, they survive for small enough M and $t_2\sin\phi$. (The direct transition from the normal to the anomalous semiconductor phase as M is varied is then replaced by an intermediate spin-polarized metallic phase.) For the realization of the internal field proposed earlier, $\gamma\hbar$ (in units of the rydberg) is given by $C'g/a^2$, where C' is another geometrical constant of order unity, and g is the Landé g factor for the electrons.

While the particular model presented here is unlikely to be directly physically realizable, it indicates that, at least in principle, the QHE can be placed in the wider context of phenomena associated with broken time-reversal invariance, and does not necessarily require external magnetic fields, but could occur as a consequence of magnetic ordering in a quasi-two-dimensional system.

This requirement is *not* fulfilled by the physical system (a domain wall in a PbTe-type semiconductor) in which Fradkin, Dagotto, and Boyanovsky[8] (FDB) have recently proposed related effects may be realized. In this model, spin-orbit coupling is supposed to give rise to the effect, but this does *not* break time-reversal symmetry. In fact, in "simplifying" the p bands of the Hamiltonian that describes PbTe, FDB introduce an unphysical effective spin-dependent hopping term that is *odd* under time reversal, and thus break the time-reversal invariance of the original physically motivated model. This, rather than any topological character of the domain wall, is the reason that FDB find the "parity anomaly" at the end of their calculation.

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