Bott clock and the Periodic Table

YuLin Qin

2024.10.25

Contents

List of Appendics

§1 The Mechanism behind the face of the Bott clock[1]

$\Re$.1 the relax of the homotopy to the reduced K theory

In physics, we consider the hamiltonian of different real dimension, if we require the translation symmetry, then we can transform the whole hamiltonian to the Fourier space and we have the BZ, in each point k of the BZ, we have H(k), and the occupied states forms a vector principle bundle over BZ, the topological classification of the quantum matter is thus reduced to the classification of such bundles.

this is achieved by the K theory. strictly speaking, we should classify such bundle under the language of homotopy, but it's much difficult and we can simplify it by relaxing the notion of equivalence. this is achieved from the reduced K theory in mathematics point of view and inspired from the physics intuition.

for example, the physical system with only one occupied band with Chern number n is not equivalent to the occupied n bands with each band Chern number equals to one, because the Fiber bundle in the previous case is rank one but the Fiber bundle in the last case is rank n, they can not equivalent in the sense of strictly homotopy. but these two system will give us the same topological quantum hall conductance in physics view of point. if we can relax the equivalence to that bundles of different ranks are counted as equivalent if we can deform them into each other after adding suitable trivial bundles. then they are equivalent, because we can add n-1 trivial bundle to the previous one and deform this to the latter one. and this kind of equivalence is called reduced K theory and such bundles over base space X is denoted as $\tilde{K}(X)$. the following classification is in the sense of $\tilde{K}(X)$, namely under the extended equivalence.

As for the base space, in physics we consider the base space to be torus, which is much complex than the sphere, in the strong topological sense(higher enough dimension), there is no difference but in the weak topological case(low space dimension), this may be different, we consider the case X to be sphere.

$\Re$.2 Bott periodic for the orthogonal group and unitary group

$\oint$.1 Altland and Zirnbauer's Approach for BdG Hamiltonian

discrete symmetries $\jmath_1, \jmath_2 \cdots$, denoted by s, when considering N dimensional system, the corresponding N dimensional representation of these symmetries are N by N complex metrics, $J_1, J_2 \cdots$. we consider the space of all N by N complex metrics representing the hamiltonian which has these symmetries(namely all the complex metrics H satisfying $[H, J_i] = 0$), this space is donated as $\mathcal{H}_s$, they find that for each given class, $\mathcal{H}_s$ is a symmetric space

$$\mathcal{H}_s = G/K$$

and the hamiltonian iH is the generator of this symmetric space. see Appendix for more details about the symmetry space.

we now show thier approaches in the following in more details to get the idea behind the construction of these symmetry space[2].

the hamiltonian of the system can be written as

$$\hat{H} = \sum_{\alpha,\beta}(h_{\alpha,\beta}c^\dagger_\alpha c_\beta + \frac{1}{2}\Delta_{\alpha,\beta}c^\dagger_\alpha c^\dagger_\beta + \frac{1}{2}\Delta^*_{\alpha,\beta}c_\beta c_\alpha)$$

$h_{\alpha,\beta} = h^*_{\beta,\alpha}$ since $\hat{H}$ should be hermitian. $\Delta_{\alpha,\beta} = -\Delta_{\beta,\alpha}$ since this system is fermion system:

$$h^\dagger = h \quad \Delta = -\Delta^T$$

we can write this hamiltonian in the BdG form with the particle-hole symmetry:

$$\hat{H} = \frac{1}{2}(c^\dagger, c)\begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}\begin{pmatrix} c \\ c^\dagger \end{pmatrix} + \text{constant}$$

so we can assign every hamiltonian the following metrics

$$\mathcal{H} = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}$$

where h is the part acting on the particle space(the state satisfying $c^\dagger|p\rangle = 0$), and $-h^T$ is the part acting on the hole space(the state satisfying $c|h\rangle = 0$). we thus introduce a particle-hole symmetry of this system, which is represented by $P := \Sigma_x := \sigma_x \otimes I_{2N}$ with σ_x act on the particle-hole space. with 2N the total dimension of h, namely the range of index α, since there are N sites and there is spin in each site. then we consider different symmetry constrains on this metrics.

Symmetry class D

since H is hermitian, which is not closed under the Lie bracket(so, it can not be the lie algebra of some metrics groups), since

$$[A, B]^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B, A] = -[A, B]$$

so we instead consider the $X := iH$, which is anti-hermitian, and closed under the Lie bracket. then the constrain on the X can be summarized as

$$-X^\dagger = X = -\Sigma_x X^T \Sigma_x$$

then such bunches of X are closed under the Lie bracket. which is isomorphic to some Lie algebra of some Lie group.In order to identify this algebra in the standard form , we can make a unitary conjugation of X by

$$-UX^\dagger U^{-1} = UXU^{-1} = -U\Sigma_x X^T \Sigma_x U^{-1} = -U\Sigma_x U^T (UXU^{-1})^T U^{-1,T}\Sigma_x U^{-1}$$

with the further requirement $U^{-1,\dagger} = U$, $U\Sigma_x U^T = I$ so as to simplify the relation. since $\Sigma_x = \sigma_x \otimes I_{2N}$, so we can write U as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes I_{2N}$ and since then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2ab & ad+bc \\ bc+ad & 2cd \end{pmatrix} = I \to ab = cd = \frac{1}{2} \;\; ad+bc = 0$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = I \to aa^* + bb^* = cc^* + dd^* = 1 \;\; ac^* + bd^* = 0$$

from $ab = cd = \frac{1}{2}$ and $aa^* + bb^* = cc^* + dd^* = 1$we have the relation that

$$aa^* = bb^* = cc^* = dd^* = \frac{1}{2} \to U = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_1} & e^{-i\phi_1} \\ e^{i\phi_3} & e^{-i\phi_3} \end{pmatrix}$$

further from $ad + bc = 0$ and $ac^* + bd^* = 0$, we have

$$\text{Re}(e^{i(\phi_1-\phi_3)}) = 0 \to \phi_1 - \phi_3 = \pm\frac{\pi}{2}$$

and we can choose that $\phi_1 = 0$ and $\phi_3 = \frac{\pi}{2}$ and we have

$$X \to \tilde{X} = U_0 X U_0^{-1} \qquad U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \otimes I_{2N}$$

then the constrain becomes

$$-\tilde{X}^\dagger = \tilde{X} = -\tilde{X}^T$$

this means that $\tilde{X}$ is real and anti-symmetric $4N \times 4N$ metrics, so it's an element of $\mathfrak{so}(4N)$. the symmetry space for this case is just

$$S_D = SO(4N)$$

from the Cartan's maximal torus theorem, we know that X can be diagonalized by the Lie group element g,namely

$$gXg^{-1} = \Omega = \sigma_z \otimes i\omega$$

which is diagonal, and g is the Lie group element defined by(which is isomorphic to SO(4N))

$$g^{-1,\dagger} = g = \Sigma_x g^{-1,T} \Sigma_x$$

then, the hamiltonian can be written as

$$\hat{H} = \frac{1}{2} \sum_\lambda \omega_\lambda (\gamma_\lambda^\dagger \gamma_\lambda - \gamma_\lambda \gamma_\lambda^\dagger)$$

we have the particle-hole operator acting on the hamiltonian as

$$P(H) = U_P H^* U_P^{-1} = -H \to U_P H^T U_P^{-1} = -H$$

so we have in terms of X=iH

$$U_P i H^T U_P^{-1} = -iH \to X = -U_P X^T U_P^{-1}$$

so in this case , the particle hole operator is just $P = \Sigma_x K$ and since then, we have

$$P = \Sigma_x K \to P^2 = \Sigma_x \Sigma_x^* = \Sigma_x^2 = +1$$

which is the same as the general definition of symmetry class D.

Symmetry class C

if we further require the system to be spin-rotation invariant, then we can derive this symmetry class. we write the particle-hole decomposition of X as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad X = E_{pp} \otimes A + E_{ph} \otimes B + E_{hp} \otimes C + E_{hh} \otimes D$$

then the constrain inherited from the $-X^\dagger = X = -\Sigma_x X^T \Sigma_x$ is that

$$-A^\dagger = A \quad C = -B^\dagger \quad B = -B^T \quad C = -C^T \quad D = -A^T$$

and then the generators of the spin rotations, are represented in this particle-hole space by

$$J_k = (E_{pp} \otimes \sigma_k - E_{hh} \otimes \sigma_k^T) \otimes 1_N$$

since these three generators form the Lie algebra of $\mathfrak{su}(2)$. then spin rotation invariant requires that

$$[X, J_k] = 0 \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \sigma_k \otimes I_N & 0 \\ 0 & -\sigma_k^T \otimes I_N \end{pmatrix} = \begin{pmatrix} \sigma_k \otimes I_N & 0 \\ 0 & -\sigma_k^T \otimes I_N \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so we have

$$[A, \sigma_k \otimes I_N] = 0 \;\; [D, \sigma_k^T \otimes I_N] = 0 \;\; -B\sigma_k^T \otimes I_N = \sigma_k \otimes I_N B \;\; C\sigma_k \otimes I_N = -\sigma_k^T \otimes I_N C$$

since $-A^\dagger = A$ we can write A as $(\lambda_i \sigma_i) \otimes a$, with λ_i be real number, so as that this constrain means that $-a^\dagger = a$.

besides $[A, \sigma_k \otimes I_N] = 0 \to [(\lambda_i \sigma_i), \sigma_k] = 0 \to \lambda_1 = \lambda_2 = \lambda_3 = 0$, which means that

$$A = \lambda_1 I \otimes a = I \otimes (\lambda_1 a) := I \otimes a$$

by means of redefine a. and $D = -A^T = I \otimes -a^T$

since $\sigma_x^T = \sigma_x, \sigma_y^T = -\sigma_y, \sigma_z^T = \sigma_z$, so if we write B as $(\lambda_i \sigma_i) \otimes b$, then $\lambda_i \sigma_i$ should commute with σ_y and anti-commute with σ_x, σ_z which means that $\lambda_i \sigma_i = \lambda_2 \sigma_y$, thus $B = \lambda_2 \sigma_y \otimes b := \sigma_y \otimes b$, the constrain $B^T = -B$ implies $b = +b^T$ and we have $C = -B^\dagger = \sigma_y \otimes -b^\dagger$. so the metrics has the following form

$$X = \begin{pmatrix} a & 0 & 0 & -ib \\ 0 & a & ib & 0 \\ 0 & ib^\dagger & -a^T & 0 \\ -ib^\dagger & 0 & 0 & -a^T \end{pmatrix} \tag{1}$$

if we absorb the complex unit i into the metrics b and define $\tilde{b} = -ib$, then the constrain on $\tilde{b}$ is also $\tilde{b}^T = b$ and we have

$$X = \begin{pmatrix} a & 0 & 0 & \tilde{b} \\ 0 & a & -\tilde{b} & 0 \\ 0 & \tilde{b}^\dagger & -a^T & 0 \\ -\tilde{b}^\dagger & 0 & 0 & -a^T \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & \tilde{b} \\ 0 & a & -\tilde{b} & 0 \\ 0 & -c & -a^T & 0 \\ c & 0 & 0 & -a^T \end{pmatrix}$$

where we have define $c = -\tilde{b}^\dagger$, we can find that in this case, the hamiltonian reduced to two commuting part, which is isomorphic, namely,(we write $\tilde{b}$ as b)

$$X = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \oplus \begin{pmatrix} a & -b \\ -c & -a^T \end{pmatrix}$$

so we can just consider one block, we consider the first block, which consists of spin-up particle and spin-down holes, we write it as

$$X_\uparrow = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$$

constrains are

$$-a^\dagger = a \quad b^T = b \quad c = -b^\dagger$$

which give constrain on the block as

$$-X_\uparrow^\dagger = X_\uparrow = -\Sigma_y X_\uparrow^T \Sigma_y \quad \Sigma_y = \sigma_y \otimes I_N$$

since the lie algebra elements of $Sp(2N) = U(2N) \cap Sp(2N, \mathcal{C})$ satisfying

$$-X^\dagger = X = JX^TJ \ \ J^2 = -1$$

then we can find that $i\Sigma_y = i\sigma_y \otimes I_N$ serve as the role of the standard J. which means that $X_\uparrow$ is the element of $\mathfrak{sp}(2N)$. so the symmetry space in this class is

$$S_C = Sp(2N) \tag{2}$$

in this case, the reduced particle-hole operator in the half block space is

$$P = \Sigma_y K \to P^2 = \Sigma_y \Sigma_y^* = -\Sigma_y^2 = -1$$

which is the sam as the general definition. we should further notice that from calls D, we add a spin-rotation symmetry, then the whole space reduced to two commuting and isomorphic block, then on just one block, the effective particle hole operator has the property of the class C. in this formalism, we derive symmetry class C from symmetry class D.

Symmetry class DIII

in this case, we should consider the time-reversal symmetry, since T act in H as

$$T(H) = U_T H^* U_T^{-1} = H \to U_T H^T U_T^{-1} = H$$

we in terms of X=iH, it should be

$$U_T iH^T U_T^{-1} = iH \to U_T X^T U_T^{-1} = X$$

so with further, time reversal symmetry, we have $U_T := \tau = I_2 \otimes i\sigma_y \otimes I_N$, where the tensor product act on PH-Spin-Sites space.

$$-X^\dagger = X = -\Sigma_x X^T \Sigma_x = \tau X^T \tau^{-1}$$

since the set satisfying the relation $X = \tau X^T \tau^{-1}$ is not closed under Lie bracket due to the simple fact that

$$[X, Y] = \tau X^T \tau^{-1} \tau Y^T \tau - \tau Y^T \tau^{-1} \tau X^T \tau = -\tau [X, Y]^T \tau$$

but we know $\tau^2 = \pm 1$, so τ is order two automorphism, so we can separate the space into the +1 eigenvalue space and -1 eigenvalue space , namely,

$$X = X_+ \oplus X_- \quad\quad X_+ = \tau X_+^T \tau^{-1} \quad\quad X_- = -\tau X_-^T \tau^{-1}$$

and the vectors with the latter property is closed under the Lie bracket

$$[X_-, Y_-] = \tau X_-^T \tau^{-1} \tau Y_-^T \tau - \tau Y_-^T \tau^{-1} \tau X_-^T \tau = -\tau [X_-, Y_-]^T \tau$$

so these vector forms a Lie algebra of some Lie group, we denoted it as the space $\mathcal{K}$, with the following constrains

$$-X^\dagger = X = -\Sigma_x X^T \Sigma_x = -\tau X^T \tau^{-1}$$

since the space of the following constrain is $\mathfrak{so}(4N)$,

$$-X^\dagger = X = -\Sigma_x X^T \Sigma_x$$

the desired space with time reversal symmetry, denoted as $\mathcal{P}$, with elements X satisfying

$$-X^\dagger = X = -\Sigma_x X^T \Sigma_x = \tau X^T \tau^{-1}$$

is the complement of $\mathcal{K}$ over $\mathfrak{so}(4N)$, namely

$$\mathfrak{so}(4N) = \mathcal{P} \oplus \mathcal{K}$$

so we only need to identify the Lie algebra $\mathcal{K}$,the equations define this space is

$$-X^\dagger = X = -\Sigma_x X^T \Sigma_x = -\tau X^T \tau^{-1}$$

since we can use time reversal and particle hole symmetry to form a unitary transformation, we can modify this by noting that

$$X^T = -\tau^{-1} X \tau = -\tau X \tau^{-1} \to -X^\dagger = X = -\Sigma_x X^T \Sigma_x = \Sigma_x \tau X (\Sigma_x \tau)^{-1}$$

we can make a unitary transformation to simplify the last relation

$$-U^{-1} X^\dagger U = U^{-1} X U = -U^{-1} \Sigma_x U^{T,-1} U^T X^T U^{-1,T} U^T \Sigma_x U = U^{-1} (\Sigma_x \tau) U U^{-1} X U U^{-1} (\Sigma_x \tau)^{-1} U$$

by require that $U^{-1} = U^\dagger$,$U^{-1} \Sigma_x U^{-1,T} = e^{i\theta} \Sigma_x$,, $U^{-1}(\Sigma_x \tau) U = e^{i\phi} \Sigma_z$ then the $\tilde{X} = U^{-1} X U$ satisfying

$$-\tilde{X}^\dagger = \tilde{X} = -\Sigma_x \tilde{X}^T \Sigma_x = \Sigma_z \tilde{X} \Sigma_z$$

where $\Sigma_x = \sigma_x \otimes I_2 \otimes I_N$,$\Sigma_z = \sigma_z \otimes I_2 \otimes I_N$. in oder to establish this , the constrain on U is(we can write U as $A \otimes B \otimes I_N$)

$$U^{-1} = U^\dagger$$
$$\Sigma_x = e^{i\theta} U \Sigma_x U^T$$
$$(\Sigma_x \tau) U = e^{i\phi} U \Sigma_z$$

if we write $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $(\Sigma_x \tau) U = e^{i\phi} U \Sigma_z$ implies that

$$i\sigma_y C = e^{i\phi} A \quad i\sigma_y D = -e^{i\phi} B \quad i\sigma_y A = e^{i\phi} C \quad i\sigma_y B = -e^{i\phi} D$$

this implies that $-e^{2i\phi} = 1$ so $e^{i\phi} = i$ and we have

$$C = \sigma_y A \quad D = -\sigma_y B$$

then $\Sigma_x = e^{i\theta} U \Sigma_x U^T$ implies that

$$AB^T + BA^T = 0 \quad CD^T + DC^T = 0 \quad AD^T + BC^T = e^{i\theta}$$

inserting into C and D by A and B, we have

$$AB^T + BA^T = 0 \quad AB^T - BA^T = e^{i\theta} \sigma_y \to AB^T = \frac{1}{2} e^{i\theta} \sigma_y$$

from $U^{-1} = U^\dagger$, we have

$$AA^\dagger + BB^\dagger = I \quad AC^\dagger + BD^\dagger = 0$$

also by inserting C and D by A and B, we have

$$AA^{\dagger} + BB^{\dagger} = I \quad AA^{\dagger} - BB^{\dagger} = 0 \to AA^{\dagger} = BB^{\dagger} = \frac{1}{2}$$

then we have $B^T = e^{i\theta} A^{\dagger} \sigma_y \to B = -e^{i\theta} \sigma_y A^*$, these are all constrains, so we can choose $A = \frac{1}{\sqrt{2}} I$ and $-e^{i\theta} = i$, then A=I and $B = \frac{1}{\sqrt{2}} i\sigma_y$, then U read as

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & i\sigma_y \\ \sigma_y & -iI_2 \end{pmatrix} \otimes I_N$$

so in conclusion, under the U conjugation, we have the equations for the elements in $\mathcal{K}$

$$-\tilde{X}^{\dagger} = \tilde{X} = -\Sigma_x \tilde{X}^T \Sigma_x = \Sigma_z \tilde{X} \Sigma_z$$

if we write $\tilde{X} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, this just means that

$$-\begin{pmatrix} A^{\dagger} & C^{\dagger} \\ B^{\dagger} & D^{\dagger} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = -\begin{pmatrix} D^T & B^T \\ C^T & A^T \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

this means that B=C=0 and $D = -A^T$ and $-A^{\dagger} = A$

$$\tilde{X} = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}$$

then it's obviously that this metrics consists of two isomorphic commuting block, and each block is determined by an anti-hermitian metrics of dimension 2N. so this Lie algebra is isomorphic to $\mathfrak{u}(2N)$, which mean that $\mathcal{K} = \mathfrak{u}(2N)$. and we have the symmetry space for this class DIII is just the space

$$S_{DIII} = SO(4N)/U(2N) \tag{3}$$

we can see that $P^2 = +1$ as before and $T^2 = i\sigma_y K i\sigma_y K = (i\sigma_y)^2 = -1$, which is the same as that in the general case.

Symmetry class CI

in this case, we proceed as adding time reversal symmetry to the spin rotation invariant system.since after considering the spin rotation invariant, the whole system reduced to two isomorphic sectors and in each sector , the Particle hole symmetry satisfying($P^2 = -1$)

$$-X_{\uparrow}^{\dagger} = X_{\uparrow} = -\Sigma_y X_{\uparrow}^T \Sigma_y$$

time reversal symmetry acting on the spin- space, and under this one sector $X_{\uparrow}$, it should be act as the identity effectively, so the constrain after considering time reversal symmetry, the whole requirement is that

$$-Y^{\dagger} = Y = -\Sigma_y Y \Sigma_y = Y^T$$

we donate the space of metrics satisfying above as $\mathcal{P}$, due to the same reason that $\mathcal{P}$ is not closed under the Lie bracket,

$$[X, Y]^T = Y^T X^T - X^T Y^T = -[X, Y] \neq [X, Y]$$

so it can not be the Lie algebra of some Lie group, instead, we consider it's complement on the Lie algebra of $\mathfrak{sp}(2N)$ which we identified before:

$$-Y^\dagger = Y = -\Sigma_y Y \Sigma_y$$

since time reversal is oder 2, the complement should be the space with the -1 eigen-value, namely, the elements in space $\mathcal{K}$ satisfying

$$-Y^\dagger = Y = -\Sigma_y Y \Sigma_y = -Y^T$$

similarly, we then try to identify this Lie algebra $\mathcal{K}$.

from $-\Sigma_y Y \Sigma_y = -Y^T$, we know that it should be commute with Σ_y, so it can be written as

$$Y = I_2 \otimes A + \sigma_y \otimes B$$

$-Y^\dagger = Y$ requires that $A^\dagger = -A, B^\dagger = -B$,$Y = -Y^T$ require that $A = -A^T, B = B^T$, so have that A is real and anti-symmetric but B is purely imaginary $B = iC$ with C symmetric

$$Y = I_2 \otimes A + \sigma_y \otimes B = Y = I_2 \otimes A + i\sigma_y \otimes C$$

since an anti-hermitian metrics X satisfying that it's real part is real anti-symmetric and it's imaginary part is real symmetric, so we have

$$Y = I_2 \otimes \mathrm{Re}(X) + i\sigma_y \otimes \mathrm{Im}(X)$$

with X anti-hermitian $(N \times N)$ metrics, so we find that Y is isomorphic to anti-hermitian metrics space by identify the new complex unit as $i\sigma_y$.

this shows that $\mathcal{K} = \mathfrak{u}(N)$. so the system with both time reversal symmetry and spin rotation symmetry is $\mathcal{P}$ satisfying:

$$\mathfrak{sp}(2N) = \mathcal{P} \oplus \mathfrak{u}(N)$$

so the symmetry space in this case is just

$$S_{CI} = Sp(2N)/U(N) \tag{4}$$

and we have the effective $P^2 = -1$ and $T^2 = +1$ in the spin up sector $X_\uparrow$, which is the symmetry class define the class CI in general case.

$\oint$.2 Symmetry space construction from the extension of Clifford Algebra

Why Clifford Algebra : The general idea and arguments on symmetry operators

For the real case, if there is no extra symmetry, we can start from a continuous space $O(16r)$, we begin from this since it's the biggest space with real entries. because the we regard the hamiltonian as the generator of the Lie groups, so the hermitian naturally requires that $X = iH$ is anti-hermitian, which can be regarded as the Lie algebra of O(16r).

then we add symmetry constrain on the hamiltonian, in the lie algebra level, it should be some metrics J_i represent the symmetry, satisfying

$$X = \mp J X^T J = \pm J X J \qquad J^2 = \pm 1$$

since $X^T = -X$, and the + sign in the first place represent the time reversal symmetry, - sign represent the particle hole symmetry. in the Lie group level, this relation should be the constrain on the metrics in O(16r) satisfying

$$O^T J O = J \to X = JXJ \quad \text{if} \quad J^2 = +1 \quad O^T J O = J \to X = -JXJ \quad \text{if} \quad J^2 = -1$$

in the following, we consider the case $J^2 = -1$ at first, then the symmetry constrain on Lie algebra level is

$$X = -JXJ \to [J, X] = 0$$

since $J^2 = -1$, the subset satisfying the above relation is closed under the Lie bracket, which means that the constrain on the Lie group level

$$O^T J O = J$$

will give us a new Lie group out of the previous one after we add a new symmetry. suppose after adding a J_1 symmetry, we derive a new Lie group G_1 by the previous argument. then we want to add a new symmetry J_2, there are two kinds of choice since J_1 is an order two mapping, namely, J_2 lie in the +1 eigen-space or -1 eigen-space of J_1

$$J_2 = \pm J_1 J_2 J_1$$

since $J_2 = \frac{1}{2}(J_2 + J_1 J_2 J_1) + \frac{1}{2}(J_2 - J_1 J_2 J_2)$ and

$$\frac{1}{2}(J_2 + J_1 J_2 J_1) = J_1 \frac{1}{2}(J_2 + J_1 J_2 J_1) J_1 \quad \frac{1}{2}(J_2 - J_1 J_2 J_1) = -J_1 \frac{1}{2}(J_2 - J_1 J_2 J_1) J_1$$

if we choose J_2 satisfying $J_2 = -J_1 J_2 J_1$, which means that J_2 lie in the Lie algebra of G_1, $J_2 \in \text{Lie}(G_1)$, which may not be in a symmetry space. we want to consider the case that the choice of J_2 Lies in a symmetry space, so we consider the case that

$$J_2 = +J_1 J_2 J_1 \to J_1 J_2 = J_1^2 J_2 J_1 = -J_2 J_1 \to \{J_1, J_2\} = 0$$

then the J_2 lies in the Lie algebra of G/G_1, which is a symmetry space, and this process is just the consideration of the Clifford Algebra extension from $\mathfrak{Cl}_{n,0}$ to $\mathfrak{Cl}_{n+1,0}$, from this extension process, we can derive the symmetry space, and consequently, the symmetry class!!!

in conclusion, we at first consider a set of symmetry operators being the basis of the Clifford Algebra, namely

$$\{J_i, J_j\} = -2\delta_{i,j} I$$

The symmetry space sequence from the extension of Clifford Algebra

1. with the above argument in mind, we consider what we derive after we add a single J_1.we can choose $J_1 = i\sigma_y \otimes I_{8r}$, then the constrain on O(16r) in the Lie group level read as

$$O^T J_1 O = J_1$$

if we write $O = A \otimes B$, this means that $A^T \otimes B^T (i\sigma_y \otimes I_{8r}) A \otimes B = i\sigma_y \otimes I_{8r}$, since $O^T O = I$, we have

$$(i\sigma_y \otimes I_{8r}) A \otimes B = A \otimes B (i\sigma_y \otimes I_{8r})$$

so we have $[i\sigma_y, A] = 0$, which means that the real metrics A should be the form $\lambda_1 I + \lambda_2 i\sigma_y$. so the group G_1 after adding J_1 is the elements of $O(16r)$ with the following form

$$(\lambda_1 I + \lambda_2 i\sigma_y) \otimes B \quad B^T B = I \quad (\lambda_1 I + \lambda_2 i\sigma_y)^T (\lambda_1 I + \lambda_2 i\sigma_y) = \lambda_1^2 + \lambda_2^2 = 1$$

so we can regard $i\sigma_y$ as the new complex unit in this space since it commutes with the elements of G_1 and square to -1, so $(\lambda_1 I + \lambda_2 i\sigma_y)$ is isomorphic to the new complex number , since then $(\lambda_1 I + \lambda_2 i\sigma_y) \otimes B$ means the complexity of the real orthogonal metrics B, which we derive the $U(8r)$ group, so we derived that $G_1 = U(8r)$,mathmaticallly

$$U(8r) \cong O(16r) \cap Sp(16r, R)$$

2. what happens if we add a new symmetry J_2, since $J_2^2 = -1$ and it anti-commute with J_1, we can write it as

$$J_2 = \sigma_z \otimes i\sigma_y \otimes I_{4r}$$

from the above argument, we can write the elements in G_1 as $(\lambda_1 I + \lambda_2 i\sigma_y) \otimes B_1 \otimes C$

$$J_2 O = OJ_2 \to (\lambda_1 I + \lambda_2 i\sigma_y) \otimes B_1 \otimes C\sigma_z \otimes i\sigma_y \otimes I_{4r} = \sigma_z \otimes i\sigma_y \otimes I_{4r}(\lambda_1 I + \lambda_2 i\sigma_y) \otimes B_1 \otimes C$$

which means that

$$(\lambda_1 I + \lambda_2 i\sigma_y)\sigma_z \otimes B_1 i\sigma_y = \sigma_z(\lambda_1 I + \lambda_2 i\sigma_y) \otimes i\sigma_y B_1$$

the solutions are

$$\lambda_1 = 0 \quad B_1 = \mu_1\sigma_x + \mu_2\sigma_z$$

$$\lambda_2 = 0 \quad B_1 = \mu_1 I + \mu_2 i\sigma_y$$

so we have the $(\lambda_1 I+\lambda_2 i\sigma_y)\otimes B_1 = \lambda_2 i\sigma_y\otimes(\mu_1\sigma_x+\mu_2\sigma_z)$ or $(\lambda_1 I+\lambda_2 i\sigma_y)\otimes B_1 = \lambda_1 I\otimes(\mu_1 I+\mu_2 i\sigma_y)$, in general, we can write it as their combinnation:

$$(\lambda_1 I + \lambda_2 i\sigma_y) \otimes B_1 = aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z$$

since the basis

$$I \otimes I, I \otimes i\sigma_y, i\sigma_y \otimes \sigma_x, i\sigma_y \otimes \sigma_z$$

form the basis of the quaternion, because the latter three squares to -1 and follow the algebra of $ij = k = -ji$ etc. so the elements of $(\lambda_1 I + \lambda_2 i\sigma_y) \otimes B_1 \otimes C$ is just the quaternization of the real orthogonal metrics C, and from $O^T O = I$, it should be the elements of $U(4r, \mathcal{H})$, where $\mathcal{H}$ represent the quaternion numbers. mathematically:

$$G_2 = U(8r) \cap Sp(8r, C) := Sp(4r) \cong U(4r, \mathcal{H})$$

3. what happens if we add a new J_3, there are different cases, which resembles the different cases of the metrics $K = J_1 J_2 J_3$, since K commute with J_1, J_2 and squares to +1. So, it can be diagonalized within the real metrics due to the fact that it's eigenvalue is either +1 or -1 which is real. the elements in G_3 must satisfying

$$O^T KO = O^T J_1 OO^T J_2 OO^T J_3 O = J_1 J_2 J_3 = K \to [K, O] = 0$$

so we can use the basis where K is diagonal, namely, the vector space $V_\pm$ which correspond to the $\pm$ eigen-space of K. then O must be block diagonal since $[K, O] = 0$, it can not change the eigen-space from V_+ to V_-. then J_1, J_2 in each eigen-space give us a quaterinion structure by requiring that

$$[O, J_1] = [O, J_2] = 0$$

so, the group which commutes with J_1, J_2, J_3, which is the same as the group which commutes with J_1, J_2, K should be the group

$$G_3 = Sp(n_1) \times Sp(n_2) \qquad n_1 + n_2 = 4r$$

in the following, we use the explicit form of J_3 to give us the concrete example of this directly. since $J_1 = i\sigma_y \otimes I_2 \otimes I_2 \otimes I_2$, $J_2 = \sigma_z \otimes (i\sigma_y) \otimes I_2 \otimes I_2$.

so one choice of J_3 is $J_3 = \sigma_x \otimes i\sigma_y \otimes I_2 \otimes I_2$. since the elements of the form $aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z \otimes B \otimes C$ is already commute with J_3, since

$$[I \otimes I, \sigma_x \otimes i\sigma_y] = 0 \quad [I \otimes i\sigma_y, \sigma_x \otimes i\sigma_y] = 0 \quad [i\sigma_y \otimes \sigma_x, \sigma_x \otimes i\sigma_y] = 0 \quad [\sigma_y \otimes \sigma_z, \sigma_x \otimes i\sigma_y] = 0$$

so in this case, the group G_3 is the same as the group $G_2 = Sp(4r)$, this is the fact that

$$K = J_1 J_2 J_3 = i\sigma_y \sigma_z \sigma_x \otimes (i\sigma_y)^2 = I \otimes I$$

only has +1 eigen-space and so that $n_1 = 4r$ and $n_2 = 0$, so that

$$G_3 = Sp(n_1) \times Sp(n_2) = Sp(4r) \times Sp(0) \cong Sp(4r)$$

another choice of J_3 is that $J_3 = \sigma_x \otimes i\sigma_y \otimes \sigma_z \otimes I_2$. then we can write the elements which commute with J_1, J_2 as $aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z \otimes B \otimes C$, the further requirement that it commute with J_3 is that

$$\begin{aligned}
&((aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes B \otimes C)(\sigma_x \otimes i\sigma_y \otimes \sigma_z \otimes I_2) \\
=&(\sigma_x \otimes i\sigma_y \otimes \sigma_z \otimes I_2)((aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes B \otimes C)
\end{aligned}$$

which means that

$$\begin{aligned}
&(a\sigma_x \otimes i\sigma_y - b\sigma_x \otimes I - c\sigma_z \otimes \sigma_z + d\sigma_z \otimes \sigma_x) \otimes B\sigma_z \\
&= (a\sigma_x \otimes i\sigma_y - b\sigma_x \otimes I - c\sigma_z \otimes \sigma_z + d\sigma_z \otimes \sigma_x) \otimes \sigma_z B
\end{aligned}$$

which mens that B is commute with σ_z,so $B = \lambda_1 I + \lambda_2 \sigma_z = \mathrm{diag}\{\mu_1, \mu_2\}$ so we have $(aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes B \otimes C$ can be written as two diagonal block with respect to B, in each block, it's a group isomorphic to $Sp(2r)$ since it's a quaternion over a real orthogonal metrics C.

so we have

$$G_3 = Sp(2r) \times Sp(2r)$$

in this case, $K = J_1 J_2 J_3 = i\sigma_y \sigma_z \sigma_x \otimes (i\sigma_y)^2 \otimes \sigma_z = I \otimes I \otimes \sigma_z$ which has equal number of $\pm$ eigen-values, so that $n_1 = n_2 = 2r$

$$G_3 = Sp(n_1) \times Sp(n_2) = Sp(2r) \times Sp(2r)$$

in the last, let's consider the case where K is not diagonal, namely, consider the case $J_3 = \sigma_z \otimes \sigma_z \otimes (i\sigma_y) \otimes I_2$ then K is

$$K = i\sigma_y \sigma_z \sigma_z \otimes (i\sigma_y \sigma_z) \otimes i\sigma_y = -i\sigma_y \otimes \sigma_x \otimes i\sigma_y$$

similarly, we have

$$((aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes B \otimes C)(\sigma_z \otimes \sigma_z \otimes (i\sigma_y) \otimes I_2)$$

$$=(\sigma_z \otimes \sigma_z \otimes (i\sigma_y) \otimes I_2)((aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes B \otimes C)$$

which means that

$$(a\sigma_z \otimes \sigma_z - b\sigma_z \otimes \sigma_x + c\sigma_x \otimes i\sigma_y - d\sigma_x \otimes I) \otimes Bi\sigma_y$$
$$= (a\sigma_z \otimes \sigma_z + b\sigma_z \otimes \sigma_x + c\sigma_x \otimes i\sigma_y + d\sigma_x \otimes I) \otimes i\sigma_y B$$

the solution is that

$$a = c = 0 \;\; B = \mu_1\sigma_x + \mu_2\sigma_z$$

$$b = d = 0 \;\; B = \lambda_1 I + \lambda_2 i\sigma_y$$

which mean that it belongs to

$$(bI \otimes i\sigma_y + di\sigma_y \otimes \sigma_z) \otimes (\mu_1\sigma_x + \mu_2\sigma_z) + (aI \otimes I + ci\sigma_y \otimes \sigma_x) \otimes (\lambda_1 I + \lambda_2 i\sigma_y)$$

so the it's an linear expansion of the following eight basis

$$I \otimes i\sigma_y \otimes \sigma_x \quad I \otimes i\sigma_y \otimes \sigma_z \quad i\sigma_y \otimes \sigma_z \otimes \sigma_x \quad i\sigma_y \otimes \sigma_z \otimes \sigma_z$$
$$I \otimes I \otimes I \quad I \otimes I \otimes i\sigma_y \quad i\sigma_y \otimes \sigma_x \otimes I \quad i\sigma_y \otimes \sigma_x \otimes i\sigma_y$$

there are two elements which square to +1, namely $I \otimes I \otimes I, i\sigma_y \otimes \sigma_x \otimes i\sigma_y$, this two also commute with all the others.the remaining six are square to -1

in order to find the two quaternion structure, we label $1 = I \otimes I \otimes I$, $e = i\sigma_y \otimes \sigma_x \otimes i\sigma_y$, and $i = I \otimes I \otimes i\sigma_y$, $j = I \otimes i\sigma_y \otimes \sigma_x$, and $k = ij = I \otimes i\sigma_y \otimes \sigma_z$, then we have

$$i\sigma_y \otimes \sigma_z \otimes \sigma_x = e \times (I \otimes -i\sigma_y \otimes -\sigma_z) = ek$$

$$i\sigma_y \otimes \sigma_z \otimes \sigma_z = e \times (I \otimes -i\sigma_y \otimes \sigma_x) = -ej$$

$$i\sigma_y \otimes \sigma_z \otimes I = e \times (I \otimes I \otimes -i\sigma_y) = -ei$$

then the above eight basis can be written as

$$j \quad k \quad ek \quad -ej$$
$$1 \quad i \quad -ei \quad e$$

so we can use the following to be the new basis

$$\frac{1}{2}(1+e) \quad \frac{1}{2}(1+e)i \quad \frac{1}{2}(1+e)j \quad \frac{1}{2}(1+e)k$$
$$\frac{1}{2}(1-e) \quad \frac{1}{2}(1-e)i \quad \frac{1}{2}(1-e)j \quad \frac{1}{2}(1-e)k$$

we choose this basis for the reason that it's easy to show that the above line commutes with the latter line since

$$(1+e)(1-e) = 1 - e^2 = I - I = 0$$

in the basis where K is diagonal, we have $e = -K$ which is diagonal with diagonal elements either 1 or -1, so $\frac{1}{2}(1+e)$ and $\frac{1}{2}(1-e)$ just means the projection to the +1 and -1 eigen-space, so they should commute from this kind of view.besides $\frac{1}{2}(1+e)$ and $\frac{1}{2}(1-e)$ are constant metrics, so we can easily see that expansion of these basis are isomorphic to the quaternions number. since the are commuting, so this group tensor product with C are isomorphic to $Sp(2r) \times Sp(2r)$ if we further consider the orthogonal condition and that C is two dimensional orthogonal metrics.

. in order to add more such J_i and proceed advance, we choose the case that $n_1 = n_2$, so the Lie group G_3 is

$$G_3 = Sp(2r) \times Sp(2r)$$

and for simplicity, we choose the J_3 to be $J_3 = \sigma_x \otimes i\sigma_y \otimes \sigma_z \otimes I_2$ so as $K = I \otimes I \otimes \sigma_z \otimes I_2$ to be diagonal, and the elements of G_3 be of the form

$$(aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes \mathrm{diag}\{\mu_1, \mu_2\} \otimes C$$

together with the condition that it's orthogonal. if we write $h = aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z$, $d = \mathrm{diag}\{\mu_1, \mu_2\}$

4. To advance further, if we add a J_4, we can consider that $L = J_3 J_4$, which commute with J_1, J_2 and anti-commute with $K = J_1 J_2 J_3$ since

$$LK + KL = J_1 J_2 J_3 J_3 J_4 + J_3 J_4 J_1 J_2 J_3 = -J_1 J_2 J_4 + J_1 J_2 J_4 = 0$$

$L^2 = -1$, then suppose v_+ is a vector with $+1$ eigen-value for K, then $KLv_+ = -LKv_+ = -Lv_+$, which means that L transfer the V_+ to V_-, since the orthogonal metrics commuting with J_1, J_2, J_3 is block diagonal in the basis $V_+ \oplus V_-$, then when further commuting with J_4, which is the same as L, which should be the form of σ_x or $i\sigma_y$ in this basis if we properly organize this basis, so we have($L^2 = -1$ and we choose the standard form of $L = i\sigma_y$)

$$L \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} L = \begin{pmatrix} -H_2 & 0 \\ 0 & -H_1 \end{pmatrix} = -\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$$

so H_1 should be the same as H_2. the the group surviving should be the diagonal elements of G_3, namely $G_4 = Sp(2r) \times I \cong Sp(2r)$.

more pratically, if we choose $J_4 = \sigma_x \otimes i\sigma_y \otimes (-\sigma_x) \otimes I_2$, Thus

$$L = (\sigma_x \otimes (i\sigma_y) \otimes \sigma_z \otimes I_2)(\sigma_x \otimes i\sigma_y \otimes (-\sigma_x) \otimes I_2) = I \otimes I \otimes i\sigma_y$$

the elements of G_3 commute with this J_4, requires that

$$\begin{aligned} &((aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes \mathrm{diag}\{\mu_1, \mu_2\} \otimes C)(\sigma_x \otimes i\sigma_y \otimes -\sigma_x \otimes I_2) \\ =&(\sigma_x \otimes i\sigma_y \otimes -\sigma_x \otimes I_2)((aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes \mathrm{diag}\{\mu_1, \mu_2\} \otimes C) \end{aligned}$$

which means that

$$\begin{aligned} &(a\sigma_x \otimes i\sigma_y - b\sigma_x \otimes I - c\sigma_z \otimes \sigma_z + d\sigma_z \otimes \sigma_x) \otimes \mathrm{diag}\{\mu_1, \mu_2\}\sigma_x \\ &= (a\sigma_x \otimes i\sigma_y - b\sigma_x \otimes I - c\sigma_z \otimes \sigma_z + d\sigma_z \otimes \sigma_x) \otimes \sigma_x \mathrm{diag}\{\mu_1, \mu_2\} \end{aligned}$$

so we must have σ_x commute with $\mathrm{diag}\{\mu_1, \mu_2\}$, which can happen when $\mu_1 = \mu_2$, so $\mathrm{diag}\{\mu_1, \mu_2\} = \mu I$ which is diagonal, then the elements of the form

$$((aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z) \otimes \mu I \otimes C)$$

consisting the group $Sp(2r)$ when consider that it's orthogonal.

finally, we have $G_4 = Sp(2r)$, anyway.

5. if we further add an extra J_5, we can consider $M = J_1J_4J_5$, which is commute with $K = J_1J_2J_3$ and squares to $+1$. so it can not mix the eigen space of K, namely V_+ and V_-, since M so we can choose the basis in V_+ and V_- in which M is aslo diagonal.

$$V_+ = V_{++} \oplus V_{+-} \qquad V_- = V_{-+} \oplus V_{--}$$

where the second sign label the eigen value of M which should be +1 or -1

since $J_2M = -MJ_2$, $J_1M = MJ_1$, so we must have $J_2V_{\pm,+} = V_{\pm,-}$, with this in mind the extra J_5 makes the the quaternions $(aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z)$ into two diagonal part according to the eigen-value of M, and they are related by J_2, this is equivalent to remove a complex structure from the quaternions, so G_5 is isomorphic to $U(2r)$

more practically, if we choose $J_5 = \sigma_z \otimes \sigma_x \otimes i\sigma_y \otimes I_2$, then the extra constrain that $OJ_5 = J_5O$ should be the constrain that

$$[aI \otimes I + bI \otimes i\sigma_y + ci\sigma_y \otimes \sigma_x + di\sigma_y \otimes \sigma_z, \sigma_z \otimes \sigma_x] = 0$$

which means that b=c=0, so the elements is of the form

$$(aI \otimes I + di\sigma_y \otimes \sigma_z) \otimes \mu I \otimes C$$

consisting the the group $U(2r)$ if we use the complex unit as $i\sigma_y \otimes \sigma_z$ and after considering the orthogonal constrain.

in conclusion, we have

$$G_5 = U(2r)$$

6. if we add an extra J_6, we can consider $N = J_2J_4J_6$, which commute with K and M and squares to $+1$, so it act within $V_{\pm,\pm}$, then we can choose the basis again which N is also diagonal, so the space can be divided into

$$V_{\pm,\pm} = V_{\pm,\pm,+} \oplus V_{\pm,\pm,-}$$

where the last sign represent the eigen-value of N. since $J_1N = -NJ_1$, so each space is related by J_1, namely, $J_1V_{\pm,\pm,+} = V_{\pm,\pm,-}$, since then, this is equivalent to remove a complex structure in G_5, which makes G_6 isomorphic to $O(2r)$

more practically, if we choose $J_6 = \sigma_x \otimes I \otimes i\sigma_y \otimes I_2$, then $N = J_2J_4J_6 = \sigma_z \otimes I \otimes \sigma_z \otimes I_2$ is diagonal.

then the extra constrain is

$$[(aI \otimes I + di\sigma_y \otimes \sigma_z) \otimes \mu I \otimes C, \sigma_x \otimes I \otimes i\sigma_y \otimes I_2] = 0$$

thus we have d=0, so the form of the elements in G_6 is

$$(aI \otimes I) \otimes \mu I \otimes C$$

which is isomorphic to $O(2r)$ when consider the orthogonal constrain. thus resulting in

$$G_6 = O(2r)$$

7. if we further add a J_7, we can consider $P = J_1 J_6 J_7$, we have

$$[P, K] = 0, [P, M] = 0, [P, N] = 0$$

and P squares to 1, so we can choose the basis where P is again diagonal, then

$$V_{\pm,\pm,\pm} = V_{\pm,\pm,\pm,+} \oplus V_{\pm,\pm,\pm,-}$$

where the last sign represent the eigen-value of P. Thus the extra constrain will put G_6 into two diagonal part, so we have

$$G_7 = O(n_1) \times O(n_2) \quad n_1 + n_2 = 2r$$

more practically, if we choose $J_7 = \sigma_z \otimes \sigma_z \otimes i\sigma_y \otimes I_2$, then this will give us no more constrain, so $G_7 = G_6 = O(2r)$, this is due to the fact that

$$N = J_1 J_6 J_7 = -I \otimes I \otimes I \otimes I$$

only has eigen-value with -1, so $n_1 = 0, n_2 = 2r$, which makes that

$$G_7 = O(0) \times O(2r) \cong O(2r)$$

another choice is $J_7 = \sigma_z \otimes \sigma_z \otimes i\sigma_y \otimes \sigma_z$, then the extra constrain is that $C = A \otimes D$, where D is metrics in r dimesion and A is 2 dimension.

$$[A, \sigma_z] = 0$$

which means that $A = \lambda_1 I + \lambda_2 \sigma_z = \text{diag}\{a, b\}$, then the elements of the form

$$\text{diag}\{a, b\} \otimes D$$

is isomorphic to the group $O(r) \times O(r)$. in this case, we have

$$N = J_1 J_6 J_7 = -I \otimes I \otimes I \otimes \sigma_z$$

which has the same number of $+1, -1$ eigen-value, so $n_1 = n_2 = r$

$$G_7 = O(n_1) \times O(n_2) = O(r) \times O(r)$$

in oder to make the whole process keeping advance, we choose this one.

8. finally, if we add an extra J_8, we can consider $Q = J_7 J_8$, which commute with K,M,N and anti-commute with P, so it maps $V_{\pm,\pm,\pm,+}$ to $V_{\pm,\pm,\pm,-}$, this means that only the diagonal entry in G_7 can survive, namely, $G_8 = O(r)$.

   more practicaly, if we choose $J_8 = \sigma_z \otimes \sigma_z \otimes i\sigma_y \otimes \sigma_x$, then the extra constrain is

$$[\text{diag}\{a, b\}, \sigma_x] = 0$$

   so we must have a=b, then the elements of the following form

$$aI \otimes D$$

   is isomorphic to $O(r)$ when considering the orthogonal constrain. what happens if we add more J_i, since $\mathfrak{Cl}_{k+8,0} \cong M(16) \otimes \mathfrak{Cl}_{k,0}$. so the extra constrain will be like just an J_i adding to $O(r)$, which will comes to a second cycle as above.

in conclusion, we have

$$\cdots O(16r) \stackrel{J_1}{\supset} U(8r) \stackrel{J_2}{\supset} Sp(4r) \stackrel{J_3}{\supset} Sp(2r) \times Sp(2r) \stackrel{J_4}{\supset} Sp(2r) \stackrel{J_5}{\supset} U(2r) \stackrel{J_6}{\supset} O(2r) \stackrel{J_7}{\supset} O(r) \times O(r) \stackrel{J_8}{\supset} O(r) \cdots \quad (5)$$

and one kind choice of such symmetry breaking operators is

$$\begin{aligned} J_1 &= i\sigma_y \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_r \\ J_2 &= \sigma_z \otimes i\sigma_y \otimes I_2 \otimes I_2 \otimes I_r \\ J_3 &= \sigma_x \otimes i\sigma_y \otimes \sigma_z \otimes I_2 \otimes I_r \\ J_4 &= \sigma_x \otimes i\sigma_y \otimes \sigma_x \otimes I_2 \otimes I_r \\ J_5 &= \sigma_z \otimes \sigma_x \otimes i\sigma_y \otimes I_2 \otimes I_r \\ J_6 &= \sigma_x \otimes I_2 \otimes i\sigma_y \otimes I_2 \otimes I_r \\ J_7 &= \sigma_z \otimes \sigma_z \otimes i\sigma_y \otimes \sigma_z \otimes I_r \\ J_8 &= \sigma_z \otimes \sigma_z \otimes i\sigma_y \otimes \sigma_x \otimes I_r \end{aligned}$$

together with the operators to separate the space into commuting blocks:

$$\begin{aligned} K &= J_1J_2J_3 = I_2 \otimes I_2 \otimes \sigma_z \otimes I_2 \otimes I_r \\ M &= J_1J_4J_5 = -I_2 \otimes \sigma_z \otimes \sigma_z \otimes I_2 \otimes I_r \\ N &= J_2J_4J_6 = \sigma_z \otimes I_2 \otimes \sigma_z \otimes I_2 \otimes I_r \\ P &= J_1J_6J_7 = -I_2 \otimes \sigma_z \otimes I_2 \otimes \sigma_z \otimes I_r \end{aligned}$$

and operators which relate two different diagonal blocks

$$\begin{aligned} L &= J_3J_4 = -I_2 \otimes I_2 \otimes i\sigma_y \otimes I_2 \otimes I_r \\ Q &= J_7J_8 = -I_2 \otimes I_2 \otimes I_2 \otimes \otimes i\sigma_y I_r \end{aligned}$$

in the above formulism , we choose a specific symmetry breaking operator and then to derive the groups with the specific symmetry constrain. At present, we try to consider another question, what's the degree of freedom of choosing these symmetry operators?

namely, if we have $J_1, J_2, \cdots, J_i$ choosing, what's the space for choosing J_{i+1} which is square to -1 and anti-commute with $J_1, J_2, \cdots, J_i$, This is equivalent to consider the extension of Clifford algebra from $\mathfrak{Cl}_{i,0}$ to $\mathfrak{Cl}_{i+1,0}$ in O(16r), we denote the space of choosing this generator J_{i+1} as R_{i+2} and the group which commute with $J_1, J_2, \cdots, J_i$ in O(16r) as G_i. . Things seem to be quite hard, since we have less knowledge about this extra generator. But it will become clear if we try to figure out this space with another isomorphic space with the help of a pre-founding specific J^0_{i+1}, namely, suppose we have already have $J^0_{i+1} \in R_{i+1}$. then we donote the group which commute with $J_1, J_2, \cdots, J_i, J^0_{i+1}$ in O(16r) as G_{i+1}. since we have

$$\forall g \in G_i, (gJ^0_{i+1}g^{-1})^2 = g(J^0_{i+1})^2g^{-1} = g(-1)g^{-1} = -1$$

$$\forall g \in G_i, gJ^0_{i+1}g^{-1}J_k + J_kgJ^0_{i+1}g^{-1}J_k = g(J^0_{i+1}J_k + J_kJ^0_{i+1})g^{-1} = g0g^{-1} = 0$$

this is to say that for all g in G_i, the conjugation of J^0_{i+1} by g lie in the space R_{i+1}, so the space R_{i+1} may be just the space of the orbital J^0_{i+1} under this conjugation, since the $G_{i+1} \subset G_i$ makes J^0_{i+1} invariant, this orbital is isomorphic to the group G_i/G_{i+1}, so we have

$$R_{i+1} = G_i/G_{i+1}$$

in symmetry space language, G_i is the isometry group of R_{i+1} and G_{i+1} is the isotropy group of the specific elements $J^0_{i+1} \in R_{i+1}$. In the following, we show that G_i/G_{i+1} is a symmetry space. so we consider the Lie algebra of G_i, donated as $\mathfrak{g}_i$, since we have for any $g \in G_i$

$$J^0_{i+1} g J^{0,-1}_{i+1} J_k = -J^0_{i+1} g J_k J^{0,-1}_{i+1} = -J^0_{i+1} J_k g J^{0,-1}_{i+1} = J_k J^0_{i+1} g J^{0,-1}_{i+1}$$

thus $J^0_{i+1} g J^{0,-1}_{i+1} \in G_i$, saying conjugation by J^0_{i+1} is an transformation on G_i, besides, $J^{0,2}_{i+1} = -1$, so this is an order two isometry, so it is an involution and the fixed group of this involution is G_{i+1}, since it commute with J^0_{i+1}, so G_i/G_{i+1} is a symmetry space (see Appendix for details if need).

besides, this involution induced an isomorphic of the Lie algebra of G_i, this algebra is divided into two parts, one of which is the +1 eigen-space of this involution:

$$J^0_{i+1} X_+ J^{0,-1}_{i+1} = X_+$$

another one is the -1 eigen-space:

$$J^0_{i+1} X_- J^{0,-1}_{i+1} = -X_+$$

we can find that X_+ is just the space of the lie algebra $\mathfrak{g}_{i+1}$ of G_{i+1}. so we have

$$\mathfrak{g}_i = \mathfrak{g}_{i+1} \oplus \mathfrak{m}_i$$

they satisfy the algebra of the tangent vectors of the symmetry space due to the fact that the they are $+1, -1$ eigen-space correspondingly:

$$[\mathfrak{g}_{i+1}, \mathfrak{g}_{i+1}] \subset \mathfrak{g}_{i+1} \qquad [\mathfrak{m}_i, \mathfrak{g}_{i+1}] \subset \mathfrak{m}_i \qquad [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{g}_{i+1}$$

so the tangent space of $R_{i+1} \cong G_i/G_{i+1}$ consisting of the elements, which belongs to the -1 eigen-space of this involution, namely, it commutes with $J_1, J_2 \cdots, J_i$ and anti-commute with J^0_{i+1}. the connected part of G_i/G_{i+1} is $\mathrm{Exp}(\mathfrak{m}_i)$ and the isomorphic map between R_{i+1} and G_i/G_{i+1} in this connected part is given by

$$g J^0_{i+1} g^{-1} = e^H J^0_{i+1} e^{-H}$$

where $H \in \mathfrak{m}_i$ which commutes with $J_1, J_2 \cdots, J_i$ and anti-commutes with J^0_{i+1}, so we have the following sequence with symmetry space labelled:

$$\begin{aligned} \cdots O(16r) \underset{R_2}{\overset{J_1}{\supset}} U(8r) \underset{R_3}{\overset{J_2}{\supset}} Sp(4r) \underset{R_4}{\overset{J_3}{\supset}} Sp(2r) \times Sp(2r) \underset{R_5}{\overset{J_4}{\supset}} Sp(2r) \\ \underset{R_6}{\overset{J_5}{\supset}} U(2r) \underset{R_7}{\overset{J_6}{\supset}} O(2r) \underset{R_0}{\overset{J_7}{\supset}} O(r) \times O(r) \underset{R_1}{\overset{J_8}{\supset}} O(r) \cdots \end{aligned} \tag{6}$$

where the lower symmetry space means that the space of choosing the above generator, for example, R_4 is the symmetry space of choosing J_3, namely, the extension from $\mathfrak{Cl}_{2,0}$ to $\mathfrak{Cl}_{3,0}$

the above is the whole story of the case where $J_i^2 = -1$, then what happens if we require $J_i^2 = +1$, in this case we use E_i to replace J_i in order to remove chaos of marks.

1. if we add a single E_1 to O(16r), since $E_1^2 = +1$, we can choose the basis to be the elements of ± 1 eigen-space $V_\pm$ of E_1, then E_1 is diagonal, so the constrain

$$OE_1 = E_1 O$$

will separate O into diagonal two blocks, so the group G_1 is

$$G_1 = O(n_1) \times O(n_2) \quad n_1 + n_2 = 16r$$

for the purpose that this sequence can advance, we choose the case where $n_1 = n_2 = 8r$, namely

$$G_1 = O(8r) \times O(8r)$$

2. if a second one E_2 goes into the problem, since $E_1E_2 = -E_2E_1$, so $E_1V_+ = V_-$, so if we further require

$$OE_2 = E_2O$$

O should be reduced to the diagonal elements of G_1, namely

$$G_2 = O(8r) \otimes I_2 \cong O(8r)$$

3. if we further add more $E_i, i > 2$, we can consider the elements $J_i = E_1E_2E_i$,which commute with both E_1 and E_2 since

$$J_iE_1 = E_1E_2E_iE_1 = -E_1E_2E_1E_i = E_1E_1E_2E_i = E_1J_i$$

$$J_iE_2 = E_1E_2E_iE_2 = -E_1E_2E_2E_i = E_2E_1E_2E_i = E_2J_i$$

further more we have the anti-commuting relations between J_j:

$$J_iJ_k = E_1E_2E_iE_1E_2E_k = E_1E_2E_1E_2E_iE_k = -E_1E_2E_kE_1E_2E_i = -J_kJ_i$$

since $J_i^2 = (E_1E_2E_i)^2 = -1$. so adding k+2 E_i's to constrain the group $O(16r)$ is equivalent to adding just k J_i's to the group O(8r), This is the result from the Clifford algebra isomorphic

$$\mathfrak{Cl}_{0,k+2} \cong \mathfrak{Cl}_{k,0} \otimes \mathfrak{Cl}_{0,2} \quad \mathfrak{Cl}_{0,2} \cong R^{2\times 2}$$

and $\mathfrak{Cl}_{0,2}$ is removed by considering E_1, E_2. since we have work out the case where $J_i^2 = -1$, so we have done all the sequence: (since we care about the case r is large enough, so to identify the symmetry space R_i, we only care about the two groups, not the dimension because we can choose proper r to make the dimension mathces)

$$\begin{aligned}\cdots O(16r) \underset{R_0}{\overset{E_1}{\supset}} O(8r) \times O(8r) \underset{R_1}{\overset{E_2}{\supset}} O(8r) \underset{R_2}{\overset{E_3}{\supset}} U(4r) \underset{R_3}{\overset{E_4}{\supset}} Sp(2r) \\ \underset{R_4}{\overset{E_5}{\supset}} Sp(r) \times Sp(r) \underset{R_5}{\overset{E_6}{\supset}} Sp(r) \underset{R_6}{\overset{E_7}{\supset}} U(r) \underset{R_7}{\overset{E_8}{\supset}} O(r) \cdots\end{aligned} \tag{7}$$

using the sequence of adding J_i and E_i, we can derive the case with general p J_i and q E_i, namely, adding a Clifford algebra $\mathfrak{Cl}_{p,q}$, we consider the special case when adding one E and one J at the same time.

since $E^2 = +1$, we can use the basis where E is diagonal, so the constrain

$$OE = EO$$

reduce the group to the diagonal blocks. secondly, since $EF = -FE$, F convert V_+ to V_-, so the constrain

$$OF = FO$$

make sure the two diagonal blocks are the same, so under considering two such symmetry operators. the group survive is the same as before but with the dimension reduced to half of the previous one, for example

$$U(2r) \to U(r)$$

This is the result of isomorphic between Clifford Algebra:

$$\mathfrak{Cl}_{p,q} \cong \mathfrak{Cl}_{p-1,q-1} \otimes \mathfrak{Cl}_{1,1} \quad \mathfrak{Cl}_{1,1} \cong R^{2\times 2}$$

since we can choose the dimension properly, so we can just say it has no effect on the group. due to this reason, the group survive after adding $\mathfrak{Cl}_{p,q}$ is the same as the group surviving after adding $\mathfrak{Cl}_{0,q-p}$ or equivalently $\mathfrak{Cl}_{p-q,0}$. if we consider the space to adding an extra negative one, namely, J_i, upon the space we derived where we have already adding p negative and q positive one.we have degree of freedom of choosing this extra lies in the space R_{p-q+2} by using the isomorphsim $\mathfrak{Cl}_{p,q} \cong \otimes^q \mathfrak{Cl}_{1,1} \otimes \mathfrak{Cl}_{p-q,0}$ and the sequence (6) or equivalently , in the language of Clifford algebra extension:

$$\mathfrak{Cl}_{p,q} \underset{R_{p-q+2}}{\overset{J_{p+1}}{\Longrightarrow}} \mathfrak{Cl}_{p+1,q}$$

if we consider to adding an extra positive one, the degree of freedom of choosing this extra positive one lies in the space R_{q-p} by using the isomorphsim $\mathfrak{Cl}_{p,q} \cong \otimes^p \mathfrak{Cl}_{1,1} \otimes \mathfrak{Cl}_{0,q-p}$ and the sequence (7) or equivalently , in the language of Clifford algebra extension:

$$\mathfrak{Cl}_{p,q} \underset{R_{q-p}}{\overset{J_{q+1}}{\Longrightarrow}} \mathfrak{Cl}_{p,q+1}$$

in the last, we consider the case with unitary group U(2r), in this case, if we add an J_1 and require

$$UJ_1 = J_1 U$$

what can survive in this case, since the complex entry of the metrics is allowed for complex unitary metrics, we can always choose the basis where J_1 is diagonal whenever $J_1^2 = \pm 1$. then the constrain will make U into two diagonal blocks, so the group survive is just

$$G_1 = U(r) \times U(r)$$

when adding an extra J_2, since $J_2 J_1 = -J_1 J_2$, the extra constrain will only make only the diagonal part allowed, since then

$$G_2 = U(r)$$

thus we get the sequence for the complex unitary group and the complex Clifford Algebra extension:

$$\cdots U(2r) \underset{C_0}{\overset{J_1}{\supset}} U(r) \times U(r) \underset{C_1}{\overset{J_2}{\supset}} U(r) \cdots \tag{8}$$

∮.3 Properties of these symmetry space

there are four main properties about these symmetry space, in this section, we try to explain them in details.

1. the symmetry spaces derived above serve as two roles, one of which is inspiring from the Altland and Zirnbauer's Approach, that is:

the hamiltonian with specific symmetry lie in the tangent space of these symmetry space.

for example, if H possess the particle-hole symmetry with $P^2 = +1$ then X=iH form the Lie algebra of $SO(4N)$, which is also an Lie algebra of the symmetry space

$$R_1 = O(4N) \times O(4N)/O(4N) \cong O(4N)$$

if H possess the particle-hole symmetry with $P^2 = +1$ and the spin rotation symmetry, then the effective spin-up block of H has particle-hole symmetry with $P^2 = -1$ and $X_\uparrow = iH_\uparrow$ forms the Lie algebra of $Sp(2N)$ which is isomorphic to the symmetry space

$$R_5 = Sp(2N) \times Sp(2N)/Sp(2N)$$

if H possess the particle-hole symmetry with $P^2 = +1$ and Time reversal symmetry $T^2 = -1$, then X=iH form the Lie algebra of $SO(4N)/U(2N)$, which is also an Lie algebra of the symmetry space

$$R_2 = O(4N)/U(2N)$$

if H possess the particle-hole symmetry with $P^2 = +1$ and Time reversal symmetry $T^2 = -1$ and spin rotation symmetry, then the effective spin up block $X_\uparrow = iH_\uparrow$, and this effective hamiltonian has symmetry with $P^2 = -1$, $T^2 = +1$ since we only consider the one spin block, and it forms the Lie algebra of $Sp(N)/U(N)$, which is the Lie algebra of the symmetry space

$$R_6 = Sp(N)/U(N)$$

2. on the other hand,

   the symmetry space serve as the classification space of the fiber bundles over the base space

   for example, the classifying space of all rank N real-vector bundles is just the real Grassmannian $G_N(R^{N+m})$, since we can use this classifying space to construct the tautological trivial bundles and any vector bundles is a pull back of this bundle, we have

$$R_0 = O(N+m)/(O(N) \times O(m)) \cong G_N(R^{N+m})$$

   so the symmetry space also serve as the classifying space of some kind of Fiber Bundles. and the classifying space is uniquely determined by Groups acting on the Fibers of these Fiber Bundles.

   in the above example, the Group acting on the Fiber is $O(N)$, and the the classifying space is uniquely determined by it as

$$R_0 = \lim_{m\to\infty} O(N+m)/(O(N) \times O(m)) := BO(N)$$

   since classification of such fiber bundles is equivalent to find the homotopy groups of the classifying space by pulling back using this homotopy group element $[X, R_s]$. so it's important to figure out these homotopy groups.

3. if X is S^n, such groups is donated by $\pi_n(R_s)$, and we have

$$\pi_n(R_i) \cong \pi_{n+1}(R_{i-1}) \tag{9}$$

this can be proved by showing that the loop space of R_i, donated as ΩR_i, which is isomorphic to R_{i+1} for any i and thus we have

$$\pi_n(R_i) \cong \pi_n(\Omega R_{i-1}) \cong \pi_{n+1}(R_{i-1})$$

in order to achieve this, we consider the element $A_i = J_i^{-1} J_{i+1}$ in the following context

$$G_{i-1} \underset{R_{i+1}}{\overset{J_i}{\supset}} G_i \underset{R_{i+2}}{\overset{J_{i+1}}{\supset}} G_{i+1}$$

since $A_i J_i = -J_i A_i$ and for k<i, we have $A_i J_k = J_k A_i$, besides, $A_i^2 = (-J_i J_{i+1})^2 = -1$, these shows that A_i square to -1 and commute with $J_k, k < i$ and anti-commute with J_i, so it's is an element of the tangent space of R_{i+1}, thus the curve

$$\gamma(t) = J_i e^{\pi A_i t} = J_i \cos(\pi t) + J_i A_i \sin(\pi t) = J_i \cos(\pi t) + J_{i+1} \sin(\pi t)$$

is a geodesic curve in the space R_{i+1} in the context of equivalence by G_i conjugation due to $J_i \in R_{i+1}$, since $\gamma(0) = J_i$ is the same as $\gamma(2) = J_i$ in the space $R_{i+1} = G_i/G_{i+1}$, and $\gamma(1) = -J_i$. so this curve can be regarded as an elements of ΩR_{i+1}. the exciting point is that $\gamma(\frac{1}{2}) = J_{i+1} \in R_{i+2}$, so for each loop, we can associate an element in R_{i+2} with it by $\gamma(\frac{1}{2})$, thus we have

$$\Omega R_{i+1} \cong R_{i+2}$$

this is much similar like that the geodesic curve connecting the north and south pole can be mapped to the equator! so we have proved

$$\Omega R_{i+1} \cong R_{i+2} \rightarrow \pi_n(R_i) \cong \pi_{n+1}(R_{i-1})$$

use this result, the Bott periodic is trivial since $R_0 = R_8$ in the limit $r \rightarrow \infty$

$$\pi_{n+8}(R_s) = \pi_n(R_{s+8}) = \pi_n(R_s) \rightarrow \pi_{n+8} O(N) = \pi_n O(N)$$

4. Following the above notation, we try to show that $Q_i \cong R_{i+2}$, where $Q_i := \{Q \in \mathrm{Tan}(R_{i+1}); Q^2 = -1\}$, if we fix a specific J_i, then the the following map

$$R_{i+2} \rightarrow Q_i : J_{i+1} \rightarrow J_i^{-1} J_{i+1}$$

give us a correspondence between this two space since Q_i consisting of the elements which commute with $J_k, k < i$, anti-commute with J_i and squares to -I, which satisfied by $J_i^{-1} J_{i+1}$.

secondly, for any $Q \in Q_i$, the following map

$$Q_i \rightarrow R_{i+2} : Q \rightarrow J_i Q$$

give us an inverse map, since $J_i Q J_k = J_i J_k Q = -J_k J_i Q$, $J_i Q J_i = -J_i J_i Q$ and $(J_i Q)^2 = -J_i^2 Q^2 = -1$ so as to make sure $J_i Q_i$ an candidate of J_{i+1}. From this above two map, we have shown that

$$Q_i \cong R_{i+2}$$

we should notice that since Q_i is just the space iH with the hamiltonian H in specific symmetry class and only has ± 1 eigen-value (flattened hamiltonian). so using this isomorphism, we can use the homotopy groups of the symmetry space to reach the goal of the topological classification of the hamiltonian with specific symmetry, namely, if the hamiltonian lies in the tangent space of R_i, then the classifying space of these hamiltonian is R_{i+1}

Finally, what remains is just to figure out the correspondence between the hamiltonian with the specific symmetry and the corresponding symmetry space whose tangent space carry on these hamiltonian. Just like the Altland and Zirnbauer's Approach gives us.

∮.4 correspondence between hamiltonian class and the associated symmetry space

In order to identify the symmetry class that the hamiltonian share, we need to multiply the factor $-i$ to the tangent space of the symmetry space, because we have noticed that the hamiltonian in physics is hermitian, which is not closed under the Lie bracket, we have already multiply i to make it anti-hermitian and since then to figure out that $X = iH$ lies in some kind of symmetry space.

the complex unit factor i should satisfy the following conditions:

1. i is an element of some real orthogonal groups, this condition arises due to the fact that the maximum space $X = iH$ without any symmetry is the Lie algebra of some orthogonal groups in the real case
2. i^2=-1, this make it a complex unit
3. i should commute with the concerning tangent space $\mathfrak{m}$ of some symmetry space, this condition arises due to the fact that i act just as a complex number in the space $\mathfrak{m}$ which is isomorphic to iH

after introducing such factor i,then we have to identify the possible symmetry operator T and C which is anti-linear with respect to this i, namely

$$Ti = -iT \qquad Ci = -iC$$

after considering all the possible anti-linear operators, we can then identify the symmetries the corresponding hamiltonian $H = -iM, M \in \mathfrak{m}$ possess!

1. we start from the space R_1, which is derived from the following chain

$$O(16r) \times O(16r) \underset{R_1}{\overset{J_0}{\to}} O(16r)$$

then the tangent space of R_1 denoted as m_{-1}, is the elements in the Lie algebra of $O(16r) \times O(16r)$(which is isomorphic to $\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$) that anti-commute with J_0

$$m_{-1} = \{X \in \mathfrak{o}(16r) \oplus \mathfrak{o}(16r); XJ_0 = -J_0X\} \cong \mathfrak{o}(16r)$$

and we have the space of the elements which belong to the +1 eigen-space of the involution J_0, denoted as h_{-1} is:

$$h_{-1} = g_0 = \{X \in \mathfrak{o}(16r) \oplus \mathfrak{o}(16r); XJ_0 = +J_0X\} \cong \mathfrak{o}(16r)$$

then we need to choose a complex unit i, if we only consider the above short chain, we argue that there is no such i as following.

from the condition 1 and 2, we know that i should be the form $i = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$ where $X_{1,2}^2 = -1, X_{1,2}^T = -X_{1,2}$, since $i^T = i^{-1} = -i$.

At present, we can choose a specific $J_0 = i\sigma_y \otimes I_{16r} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, then $m_{-1} = \sigma_z \otimes X, X \in \mathfrak{o}(16r)$. then i should be commute with $\sigma_z \otimes \mathfrak{o}(16r) \cong \mathfrak{o}(16r)$, from Schular's lemma, this can happen only when $i = \lambda I$, thus $i^2 = \lambda^2 \neq -1$ within the real metrics, so no such i can exists in this short chain.

we can choose the demanded i by considering the group $O(16r) \times O(16r)$ is inherited from the previous chain, namely

$$\cdots Sp(32r) \underset{R_{-2}}{\overset{J_{-3}}{\rightarrow}} U(32r) \underset{R_{-1}}{\overset{J_{-2}}{\rightarrow}} O(32r) \underset{R_0}{\overset{J_{-1}}{\rightarrow}} O(16r) \times O(16r) \underset{R_1}{\overset{J_0}{\rightarrow}} O(16r)$$

the complex unit should be regarded as an element in $O(32r)$(which may be considered as inherited from an even larger space if need), in this longer chain, we can choose

$$i = J_{-1}$$

since $(J_{-1})^2 = -1$ and J_{-1} commute with Lie$(O(16r) \times O(16r))$, then the hamiltonian is read as

$$H = -J_{-1}m_{-1}$$

the anti-linear maps, which may be a candidate of T or C, can be choosen as $\phi = J_0$,and we find that

$$\phi H = J_0(-J_{-1})m_{-1} = J_{-1}J_0m_{-1} = -J_{-1}m_{-1}J_0 = H\phi$$

since J_0 anti-commute with m_{-1} and J_{-1} commute with it.

then this ϕ defines a Time-Reversal symmetry on the hamiltonian $H = -im_{-1} = -J_{-1}m_{-1}$ which squares to -1, how about other possible symmetries? if we choose $\phi = J_0J_{-1}$ which is anti-linear, and

$$\phi H = J_0J_{-1}(-J_{-1}m_{-1}) = -J_{-1}m_{-1}J_0J_{-1} = H\phi$$

and $\phi^2 = -1$, so it's another Time-Reversal symmetry on the hamiltonian $H = -im_{-1} = -J_{-1}m_{-1}$ which squares to -1. but this one is equal to the previous one since J_{-1} is just the complex unit i. so in conclusion, in this case, it belongs to the symmetry class C which is not coincidental with the results of their[1]. the reason lies in the fact that in the chain, we regard the symmetry breaking operator $J_{-2,-1,0}$ as the elements of $O(256r)$, which may be not effective square to -1 in the subspace R^{32r}, for example, following the notation of previous sections, $J_{-1} = J_7 \otimes I_{16r} = \sigma_z \otimes \sigma_z \otimes i\sigma_y \otimes \sigma_z \otimes I_{16r}$ which squares to -1, but in the space R^{32r} it effective acts as $\sigma_z \otimes I_{16r}$, so it squares to $+1$ in this subspace, or equivalently, only with $E_{-1}^2 = +1$, the following chain is correct

$$O(32r) \underset{R_0}{\overset{E_1}{\rightarrow}} O(16r) \times O(16r) \underset{R_1}{\overset{E_2}{\rightarrow}} O(16r)$$

which makes it an illegal complex unit i, so the above argument lose it's meaning! in order to make it legal, we need to consider even longer chain.

$$O(256r) \cdots \underset{R_{-3}}{\overset{J_{-4}}{\rightarrow}} Sp(32r) \underset{R_{-2}}{\overset{J_{-3}}{\rightarrow}} U(32r) \underset{R_{-1}}{\overset{J_{-2}}{\rightarrow}} O(32r) \underset{R_0}{\overset{J_{-1}}{\rightarrow}} O(16r) \times O(16r) \underset{R_1}{\overset{J_0}{\rightarrow}} O(16r)$$

which resembles the Clifford algebra isomorphic

$$\mathfrak{Cl}_{6,0} \rightarrow \mathfrak{Cl}_{7,0} \cong \mathfrak{Cl}_{2,0} \otimes \mathfrak{Cl}_{0,2} \otimes \mathfrak{Cl}_{2,0} \rightarrow \mathfrak{Cl}_{0,2} \otimes \mathfrak{Cl}_{2,0} \otimes \mathfrak{Cl}_{0,1} \cong R^{8\times 8} \rightarrow R^{8\times 8} \otimes \mathfrak{Cl}_{0,1}$$

which means that adding the 7-th J_7 is the same as adding just 1st E_1 to the existing system. due to the above reasons, we must consider even longer chain to find the suitable i.

but why so bother, what we need is clear, a complex unit i, we can simply make an artificial block

$$m_{-1} \to I_2 \otimes m_{-1}$$

and choose an operator that squares to -1 on considering this artificial block

$$i_n = i\sigma_y \otimes I$$

then it naturally commute with $\tilde{M} = I_2 \otimes M, M \in m_{-1}$, which make it a legal complex unit i_n(in the following we use the subscript n to distinguish between the new legal complex unit in the hamiltonian space and the standard one which is denoted as i), so the hamiltonian becomes

$$H = -i_n\tilde{M} = -(i\sigma_y \otimes I)I_2 \otimes M = -i\sigma_y \otimes M$$

since we use a new strategy to choose new complex unit i_n, so there is no other constrain on M besides that it belongs to $\mathfrak{o}(16r) \cong \mathrm{Tan}(O(16r) \otimes O(16r)/O(16r))$, so all the anti-linear operators with respect to this complex unit i_n are

$$\sigma_x \otimes I \qquad \sigma_z \otimes I$$

but they are not independent since

$$\sigma_x \otimes I = (\sigma_z \otimes I)(i\sigma_y \otimes I) = \sigma_z \otimes I i_n$$

so we can only consider one of it namekly $\phi = \sigma_z \otimes I$, then we have

$$\phi H = -H\phi$$

so this anti-linear operator defines a particle-hole symmetry on the hamiltonian which squares to $+1$, thus it belongs to the hamiltonian class D.

$$D \equiv R_1 = O(16r) \times O(16r)/O(16r) \cong O(16r) \tag{10}$$

2. as for the symmetry space R_2, there is an extra constrain on $M \in m_0$ that anti-commute with J_1

$$O(16r) \xrightarrow[R_2]{J_1} U(8r)$$

since J_1 anti-commute with $M \in m_0$, it can not also serve as an complex unit i, due to the same reason above, we use the strategy of constructing an artificial block

$$m_0 \to I_2 \otimes m_0 \qquad i_n \to i\sigma_y \otimes I \qquad \phi = \sigma_z \otimes I$$

in this case, ϕ is also a particle-hole operator which squares to -1

$$\phi H = -H\phi \qquad \phi^2 = +1$$

but, we have another anti-linear operator $\phi_1 = \phi(I_2 \otimes J_1) = \sigma_z \otimes J_1$, since $\phi_1 i_n = -i_n\phi_1$, besides, we have

$$\phi_1 H = \phi_1 i\sigma_y \otimes M = (\sigma_z \otimes J_1)(i\sigma_y \otimes M) = (i\sigma_y \otimes M)(\sigma_z \otimes J_1) = H\phi_1$$

since σ_z anti-commute with σ_y and J_1 anti-commute with M. so ϕ_1 is an Time-Reversal operator which squares to $(\sigma_z \otimes J_1)^2 = -I$, which make the hamiltonian belongs to the class DIII.

$$DIII \equiv R_2 = O(16r)/U(8r) \tag{11}$$

3. let's move on to the symmetry space R_3:

$$O(16r) \underset{R_2}{\overset{J_1}{\to}} U(8r) \underset{R_3}{\overset{J_2}{\to}} Sp(4r)$$

in this case, since J_1 commutes with the element $M \in m_1$ and squares to -1, so it can be used as an legal complex unit i_n in the hamiltonian space $H = -iM = -J_1M$. then there is only one independent anti-linear operator $\phi = J_2$, since $J_2 i_n = J_2 J_1 = -J_1 J_2$ and we have

$$\phi H = J_2 J_1 M = -J_1 J_2 M = J_1 M J_2 = H\phi$$

so ϕ is a Time-reversal Operator which squares to -1, due to this reason, the hamiltonian belongs to the symmetry class AII.

$$AII \equiv R_3 = U(8r)/Sp(4r) \tag{12}$$

4. as for the symmetry space R_4:

$$O(16r) \underset{R_2}{\overset{J_1}{\to}} U(8r) \underset{R_3}{\overset{J_2}{\to}} Sp(4r) \underset{R_4}{\overset{J_3}{\to}} Sp(2r) \times Sp(2r)$$

in this case $K = J_1 J_2 J_3$ square to +1, but M anti-commute with K, so we can not use K to put M into diagonal blocks. so we can not separate it into small blocks

in this case $M \in m_2$, which commute with J_1, J_2, so we can choose either be the legal complex unit , we choose $i_n = J_1$ as before.

then the possible independent anti-linear operators are J_2, J_3, as for $\phi_1 = J_2$, we have

$$\phi_1 H = J_2 J_1 M = -J_1 J_2 M = -J_1 M J_2 = -H\phi$$

so ϕ_1 is a Particle-Hole symmetry operator which squares to -1.

as for $\phi_2 = J_3$, we have

$$\phi_2 H = J_3 J_2 M = -J_2 J_3 M = J_2 M J_3 = H\phi_2$$

so ϕ_2 is a Time-Reversal Operator which squares to -1.$\phi_1\phi_2 = J_2 J_3$ is a linear operator and serve as the role of chiral operator which squares to -1, this also verifies that H can not be split into small blocks. thus the hamiltonian belongs to the symmetry class CII

$$CII \equiv R_4 = Sp(4r)/Sp(2r) \times Sp(2r) \tag{13}$$

5. let's move onto the symmetry space R_5:

$$O(16r) \underset{R_2}{\overset{J_1}{\to}} U(8r) \underset{R_3}{\overset{J_2}{\to}} Sp(4r) \underset{R_4}{\overset{J_3}{\to}} Sp(2r) \times Sp(2r) \underset{R_5}{\overset{J_4}{\to}} Sp(2r)$$

in this case, $K = J_1 J_2 J_3$ squares to +1 and commute with $M \in m_3$ since M commute with J_1, J_2, J_3 and anti-commute with J_4 , so M can be put into diagonal blocks with respect to the eigen-values of K, we choose the block where K equals to +1. and denoted the new block as m_3^+ since we only care about the symmetries in the irreducible blocks.

since J_1, J_2, J_3 commute with K, commute with m_3 and square to -1, so they are all legal complex unit in the hamiltonian space $J_1 m_3$ and the chosen block $J_1 m_3^+$, we choose $i_n = J_1$ as before.

in the chosen block $J_1 m_3^+$, only the operator commute with K are allowed, so all the possible anti-linear operators are J_2, J_3, since $K = J_1 J_2 J_3 = +1$ in this block, they are not independent.

$$J_2 = J_1 J_1 J_2 J_3 J_3 = J_1 + 1 J_3 = i_n J_3$$

so only one an-ti linear independent operator $\phi = J_2$

$$\phi H = J_2 J_1 M^+ = -J_1 J_2 M^+ = -J_1 M^+ J_2 = -H\phi$$

thus it serve as a Particle-Hole symmetry which squares to -1. so the chosen block hamiltonian belongs to the symmetry class C.

$$C \equiv R_5 = Sp(2r) \times Sp(2r)/Sp(2r) \cong Sp(2r) \tag{14}$$

if we consider the whole space instead of one block. then all the possible anti-linear operators are

$$J_2, J_3, J_4, J_2 J_3 J_4$$

J_2, J_3 are two Particle-Hole symmetry operator that squares to -1, and J_4 is an Time-Reversal symmetry that squares to -1, $J_2 J_3 J_4$ is an Time-Reversal Operator that squares to +1. all these symmetry reduced to one Particle-Hole symmetry operator that squares to -1 effectively in one irreducible block.

6. as for the symmetry space R_6

$$O(16r) \underset{R_2}{\overset{J_1}{\rightarrow}} U(8r) \underset{R_3}{\overset{J_2}{\rightarrow}} Sp(4r) \underset{R_4}{\overset{J_3}{\rightarrow}} Sp(2r) \times Sp(2r) \underset{R_5}{\overset{J_4}{\rightarrow}} Sp(2r) \underset{R_6}{\overset{J_5}{\rightarrow}} U(2r)$$

in this case, we at first try to figure out the possible diagonal blocks. we can try to find the operators that are mutually commute and each commutes with $M \in m_4$ and squares to +1.

in this case, $K = J_1 J_2 J_3$ is such an operator. in order to choose another one, we can only pick one of $J_1 J_2 J_3$ so as to make these two mutually commute, thus the only possibility is $M_0 = J_1 J_4 J_5$(we put the subscript 0 to avoid misleading), but M_0 anti-commute with $M \in m_4$ since J_5 anti-commute with it and J_1, J_4 commute with it

so, only, two blocks which is the ±1 eigen-space of K, as before, we consider the +1 block, namely $M^+ \in m_4^+$

and we choose the legal complex unit $i_n = J_1$ as before. then the allowed anti-linear operators in this block are(commute with $K = J_1 J_2 J_3$ and anti-commute with J_1)

$$J_2, J_3, J_2 J_4 J_5, J_3 J_4 J_5$$

and the independent anti-linear operators in this block are(note that $J_2 = i_n J_3$)

$$J_2, J_2 J_4 J_5$$

as for $\phi_1 = J_2$,we have

$$\phi_1 H = J_2 J_1 m_4^+ = -J_1 J_2 m_4^+ = -J_1 m_4^+ J_2 = -H\phi_1$$

thus ϕ_1 is a Particle-Hole symmetry operator that squares to -1 as for $\phi_2 = J_2J_4J_5$,we have

$$\phi_2 H = J_2J_4J_5J_1m_4^+ = J_1m_4^+J_2J_4J_5$$

thus ϕ_2 is a Time-Reversal symmetry operator that squares to +1

so the hamiltonian belongs to the symmetry class CI

$$CI \equiv R_6 = Sp(2r)/U(2r) \tag{15}$$

similarly, if we consider the whole space m_4, then the allowed anti-linear operators are

$$J_2, J_3, J_4, J_5, J_3J_4J_5, J_2J_4J_5, J_2J_3J_4$$

J_2, J_3, J_4 are particle-hole symmetry operators that squares to -1,$J_2J_3J_4$ is particle-hole symmetry operators that squares to +1,$J_3J_4J_5$, $J_2J_4J_5$ are time-reversal operators that square to +1, all these symmetry reduced to a Particle-Hole symmetry operator that squares to -1 and a Time-Reversal symmetry operator that squares to +1 in the diagonal one block.

7. let's go advance for the symmetry space R_7

$$O(16r) \underset{R_2}{\overset{J_1}{\rightarrow}} U(8r) \underset{R_3}{\overset{J_2}{\rightarrow}} Sp(4r) \underset{R_4}{\overset{J_3}{\rightarrow}} Sp(2r)\times Sp(2r) \underset{R_5}{\overset{J_4}{\rightarrow}} Sp(2r) \underset{R_6}{\overset{J_5}{\rightarrow}} U(2r) \underset{R_7}{\overset{J_6}{\rightarrow}} O(2r)$$

in this case, $K = J_1J_2J_3$,$M_0 = J_1J_4J_5$ mutually commute and each of it commute with $M \in m_5$ since J_1, J_2, J_3, J_4, J_5 commute with M and J_6 anti-commute with M. so we can choose the block with K, M_0 take the value +1, namely, in the space $M^{++} \in m_5^{++}$. in this case J_1 commutes with both K and M_0, besides it also commutes with M^{++} and square to -1. so it's also an legal complex unit in the hamiltonian space $J_1m_5^{++}$.

then all the possible anti-linear operators in this block are(commute with K and M_0, anti-commute with J_1)

$$J_2J_4J_6, J_2J_5J_6, J_3J_4J_6, J_3J_5J_6$$

and since $K = J_1J_2J_3 = +1$ and $M_0 = J_1J_4J_5 = +1$, so the $J_2 = J_1J_3 = i_nJ_3$ and $J_4 = J_1J_5 = i_nJ_5$, which means J_2 and J_3 are the same and J_4 and J_5 are the same, thus the independent anti-linear operator in this block is just

$$J_2J_4J_6$$

thus $\phi = J_2J_4J_6$, we have

$$\phi H = J_2J_4J_6J_1m_5^{++} = J_1m_5^{++}J_2J_4J_6 = H\phi$$

thus it defines a Time-Reversal symmetry operator that squares to +1, so the hamiltonian belongs to the symmetry class AI.

$$AI \equiv R_7 = U(2r)/O(2r) \tag{16}$$

similarly, we can discuss all the possible symmetry operators in the whole space J_1m_5, we ignore it here since it give us no more information and we only care about the irreducible blocks.

8. as for the symmetry class R_8,

$$O(16r) \underset{R_2}{\overset{J_1}{\rightarrow}} U(8r) \underset{R_3}{\overset{J_2}{\rightarrow}} Sp(4r) \underset{R_4}{\overset{J_3}{\rightarrow}} Sp(2r)\times Sp(2r) \underset{R_5}{\overset{J_4}{\rightarrow}} Sp(2r) \underset{R_6}{\overset{J_5}{\rightarrow}} U(2r) \underset{R_7}{\overset{J_6}{\rightarrow}} O(2r) \underset{R_8}{\overset{J_7}{\rightarrow}} O(r)\times O(r)$$

then $K = J_1J_2J_3, M_0 = J_1J_4J_5, N = J_2J_4J_6$ are mutually commuting and each of these commute with $M \in m_6$.thus we can just consider one block denoted as $M^{+++} \in m_6^{+++}$. but this will bring us another problem that is the choice of the complex unit i_n.

because i_n should be commute with K, M_0, N, m_6^{+++} and squares to -1. we argue that there is no such element in this block. this is the similar as the case we discussed in R_1.

at first, since J_7 anti-commute with m_6^{+++}, so, i_n can not contain any factor of J_7,since it squares to -1, so it must consist of 1,2,5,6 of such J_i's, let's enumerate it as following.

at first , just one such J_i involved, it should be commuting with K, M_0, thus it must be J_1, but J_1 is not commuting with N, so this case is excluded.

two kind of such J_i involved, so it should contain two or none factor of K, M_0 or N, that is possible, since K, M_0 or N only has at most one common factor

five kind of such J_i, thus it can be write as $\pm J_1J_2J_3J_4J_5J_6J_i$, since $J_1J_2J_3J_4J_5J_6$ anti-commutes with K, M_0, N, thus the extra J_i must anti-commute with all of K, M_0, N, that is impossible, since K, M_0, N contains one such factor J_i.

six kind of such J_i, the only possibility is $J_1J_2J_3J_4J_5J_6$ but it anti-commutes with K, M_0, N

so if we choose this block, we can not find a complex structure i_n, thus it's impossible for us to define the anti-linear operator.

instead, we choose some larger block, namely, we add the block where N equals to -1 to the hamiltonian, or equivalently, we consider the block m_6^{++} as in the case of R_7, where K and M_0 take the value of $+1$

then we can also choose the complex unit as $i_n = J_1$ as before. then the possible independant anti-linear operators are(anti-commute with J_1 and commuting with K, M_0)

$$J_2J_4J_6, J_2J_4J_7$$

if we choose $\phi_1 = J_2J_4J_6$, we have

$$\phi H = J_2J_4J_6J_1m_6^{++} = -J_1m_6^{++}J_2J_4J_6 = -H\phi$$

thus it's a Particle-Hole symmetry operator that squares to $+1$

if we choose $\phi_1 = J_2J_4J_7$, we have

$$\phi H = J_2J_4J_7J_1m_6^{++} = -J_1J_2J_4J_7m_6^{++} = +J_1m_6^{++}J_2J_4J_7 = H\phi$$

thus ϕ_2 is a Time-Reversal operator that squares to $+1$, so the hamiltonian belongs to the symmetry class BDI

$$BDI \equiv R_8 = O(2r)/O(r) \times O(r) \tag{17}$$

9. Finally, let's consider what happens if we consider one more such chains, namely R_9

$$O(16r) \xrightarrow[R_2]{J_1} U(8r) \xrightarrow[R_3]{J_2} Sp(4r) \xrightarrow[R_4]{J_3} Sp(2r) \times Sp(2r) \xrightarrow[R_5]{J_4} Sp(2r) \xrightarrow[R_6]{J_5} U(2r) \xrightarrow[R_7]{J_6} O(2r) \xrightarrow[R_8]{J_7} O(r) \times O(r) \xrightarrow[R_9]{J_8} O(r)$$

in this case, the mutually maximum commuting operators that square to $+1$ and each commute with m_7 are

$$K = J_1J_2J_3, M_0 = J_1J_4J_5, N = J_2J_4J_6, P = J_1J_6J_7$$

so we can choose one block, m_7^{++++}, due to the same reason that we can not choose a legal complex unit in this block, we must add the -1 block of the N operator to our hamiltonian, and effectively, consider the block m_7^{+++}, where K, M_0, P take the value +1.

in this case, we can still regard J_1 as the complex unit i_n, then the independent anti-linear operators in this block(commute with K, M_0, P,anti-commute with J_1) are(J_8 is not allowed in this block since it anti-commute with K, M_0, P)

$$J_2J_4J_6$$

thus $\phi = J_2J_4J_6$,and we have

$$\phi H = J_2J_4J_6J_1m_7^{+++} = -J_1J_2J_4J_6m_7^{+++} = -J_1m_7^{+++}J_2J_4J_6 = -H\phi$$

thus ϕ defines a particle hole symmetry operator that squares to +1, so the hamiltonian of the concerning block belongs to the class D. in this case, J_1 plays the role of the $i\sigma_y \otimes I$ in R_1 and $N = J_2J_4J_6$ plays the role of $\sigma_z \otimes I$ in R_1 in our artificial construction of the complex unit i_n and argue that it should be lying in some ever larger space(longer chain).

in this stage, we point out that that's just the case and we should consider the whole chain to figure out this artificial i_n, but the effect is the same, and it bring us to a full circle.

$$D \equiv R_9 = O(r) \times O(r)/O(r) \cong R_1 = O(16r) \times O(16r)/O(16r)$$

in conclusion, we collect all the results above in the following table and the following chain:

$$O(16r) \underset{R_2}{\overset{J_1}{\to}} U(8r) \underset{R_3}{\overset{J_2}{\to}} Sp(4r) \underset{R_4}{\overset{J_3}{\to}} Sp(2r) \times Sp(2r) \underset{R_5}{\overset{J_4}{\to}} Sp(2r) \underset{R_6}{\overset{J_5}{\to}} U(2r) \underset{R_7}{\overset{J_6}{\to}} O(2r) \underset{R_8}{\overset{J_7}{\to}} O(r) \times O(r) \underset{R_9}{\overset{J_8}{\to}} O(r)$$

Table 1: the symmetry space and the corresponding hamiltonian and their symmetry class

Class	T	T^2	C	C^2	Complex Unit i	$M = G_i/G_{i+1}$	H vs $m = TM$	$Q = \text{Flatten}(H)$
D	0	0	$\sigma_z \otimes I$	+	$i\sigma_y \otimes I$	$O(16r) \times O(16r)/O(16r) \cong O(16r)$	$H = -iI_2 \otimes m$	R_2
DIII	$\sigma_z \otimes J_1$	−	$\sigma_z \otimes I$	+1	$i\sigma_y \otimes I$	$O(16r)/U(16r) \cong O(16r)$	$H = -iI_2 \otimes m$	R_3
AII	J_2	−	0	0	J_1	$U(8r)/Sp(4r)$	$H = -im$	R_4
CII	J_3	−	J_2	−	J_1	$Sp(4r)/Sp(2r) \times Sp(2r)$	$H = -im$	R_5
C	0	0	J_2	−	J_1	$Sp(2r) \times Sp(2r)/Sp(2r)$	$H = -im^+$	R_6
CI	$J_2J_4J_5$	+	J_2	−	J_1	$Sp(2r)/U(2r)$	$H = -im^+$	R_7
AI	$J_2J_4J_6$	+	0	0	J_1	$U(2r)/O(2r)$	$H = -im^{++}$	$R_8 \cong R_0$
BDI	$J_2J_4J_7$	+	$J_2J_4J_6$	+	J_1	$O(2r)/O(r) \times O(r)$	$H = -im^{++}$	R_1
D	0	0	$J_2J_4J_6$	+	J_1	$O(r) \times O(r)/O(r) \cong O(r)$	$H = -im^{+++}$	R_2

Appendix

Appendix §1 Riemannian Symmetric Space

$\Re$.1 Definitions and Basic Properties

定义 §1.0 A (Riemannian) symmetric space is a Riemannian manifold S with the property that the geodesic reflection at any point is an isometry of S. Namely, for any $x \in S$, there is some $s_x \in I(S)$(the isometry group of S) with the following properties:

$$s_x(x) = x \;\; ds_x|_x = -I$$

where s_x called the symmetry at x and $ds_x|_x$ means that the map of the tangent space induced by s_x, namely

$$ds_x|_x(X) = \frac{ds_x(\gamma(t))}{dt}|_{t=0} \qquad \dot{\gamma}(t)|_0 = X \;\; \gamma(0) = x$$

since we can reflect the geodesic, so we can extend any geodesic defined in a small interval to the whole space, due to the same reason, any two endpoints of a geodesic can be mapped to each other by the reflection of the midpoint. so the isometry group of S denoted by G=I(S) act transitively on the symmetry space S.

if we fix a base point $p \in S$, the closed subgroup of G with the property $g(p) = p$ forms a group, denoted by G_p, this is called the isotropy group, we use $K = G_p$ to represent it.

suppose s_p is a symmetry at p, then for any $q = gp \in S$, we find that $s_{gp} := gs_pg^{-1}$ is a symmetry in q since

$$gs_pg^{-1}q = q \;\; dgs_pg^{-1}|_q = gds_p|_pg^{-1} = g(-I)g^{-1} = -I$$

so we have the following theorem

Theorem §1.0 A symmetric space S is precisely a homogeneous space(I(S) act on S transitively) with a sym-metry s_p at some point p S.

on the other hand, the G action is equivariant over the following map

$$G/K \to S \qquad gK \to gp$$

so we can identify the symmetry space as $S = G/K$.

$\Re$.2 Examples

$\oint$.1 the Euclidean Space

the symmetry at x is $s_x(x + v) = x - v$ since $s_x(x) = x$ and $ds_x|_x(v) = \frac{ds_x(x+vt)}{dt} = \frac{d(x-vt)}{dt} = -v$ which means that $ds_x|_x = -I$. in this case G is the euclidean group E(n) generated by translations and orthogonal linear maps; the isotropy group of the origin O is the orthogonal group O(n) and we have

$$R^n = E(n)/O(n)$$

$\oint$.2 The Sphere

if we regard S^n as the subspace of R^{n+1} with $\sum_{i=1}^{n+1} x_i^2 = 1$ with the metric induced from the standard scalar product in R^{n+1}, then the symmetry at $x \in S^n$ is the reflection along $\mathcal{R}x$, namely (note that $e_x = x$)

$$s_x(y) =< y, e_x > e_x - (y- < y, e_x > e_x) = -y + 2 < y, x > x$$

since $s_x(x) = x$ and $ds_x|_x(v) = \frac{d}{dt}(-\gamma(t) + 2 < \gamma(t), x > x)|_{t=0} = -v + 2 < v, x > x = -v$ due to the fact that $< v, x >= 0$, since we have $< \gamma(t), \gamma(t) >= 1$, so the tangent space at x is $T_xS^n = \{v \in R^{n+1}; < x, v >= 0\}$.

the isometry group is the group G=O(n+1), and the isotropy group at the point $e_{n+1} = (0, 0, \cdots, 0, 1)$ is the group $O(n)$, so we have

$$S^n = O(n+1)/O(n)$$

$\oint$.3 The Hyperbolic Space

consider the metric defined in R^{n+1} by $(x, y) = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$, the Hyperbolic Space is defined as one sheet of the Hyperbolic sphere, namely by, $H^n = \{x \in R^{n+1}; (x, x) = -1, x_{n+1} > 0\}$

in order to make this space a Riemannian manifold, we should show the tangent space is positive definite, namely $T_xH^n = \{v \in R^{n+1}; (x, v) = 0\}$ has length larger than zero.

$$\begin{aligned}(v, v) = \sum_{i=1}^{n} v_i^2 - v_{n+1}^2 &= \sum_{i=1}^{n} v_i^2 - \frac{1}{\sum_{i=1}^{n} x_i^2 + 1}(\sum_i x_i v_i)^2 \\ &> \sum_{i=1}^{n} v_i^2 - \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2 + 1}(\sum_i v_i^2) = \frac{\sum_i v_i^2}{\sum_{i=1}^{n} x_i^2 + 1} > 0\end{aligned}$$

the symmetry at point x is also the reflection along $\mathcal{R}x$, suppose we decompose y as the one along x and vertical to x with respect to the scalar product, $y = \lambda_y x + v_y$ with $(v_y, x) = 0$, then we have $(y, x) = \lambda_y (x, x) = -\lambda_y \to \lambda_y = -(y, x)$, so we have

$$s_x(y) = -(y, x)x - [y + (y, x)x] = -y - 2(y, x)x$$

we can verify this is symmetry at x explicitly

$$s_x(x) = -x + 2x = x \quad ds_x|_x(v) = -v$$

and the isometry group is the lorentz group $G = O(n, 1)^+$(+ sign due to the fact that we only pick up one sheet), the isotropy group of e_{n+1} is again O(n), so we have

$$H^n = O(n, 1)^+/O(n)$$

$\oint$.4 The Orthogonal Group

The Riemannian metric on O(n) is induced from the trace scalar product on $\mathcal{R}^{n\times n}$, namely

$$< x, y >:= \text{tr}(x^T y) = \sum_{i,j} x_{i,j} y_{i,j}$$

the left and right multiplication with orthogonal matrices preserve this inner product and make the whole space $O(n)$ invariant, so they act as isometries on O(n) turning O(n) into a homogeneous space.

besides, consider the following linear map

$$s_I(x) = x^T$$

it also preserve the inner product $< s_I(x), s_I(y) >= \text{tr}(xy^T) = \text{tr}(x^T y) =< x, y >$ and make the whole space $O(n)$ invariant. so it's also an isometry.

so the isometry group is $G =< s_I, L_g, R_{g'} >$, the isotropy group at the identity I is $K =< s_I, L_g \circ R_{g^T} >$, and we have

$$O(n) =< s_I, L_g, R_{g'} > / < s_I, L_g \circ R_{g^T} >$$

and the symmetry at the identity is just the map s_I since we have

$$s_I(I) = I^T = I \quad ds_I|_I(v) = \frac{d}{dt} s_I(\gamma(t))|_{t=0} = \frac{d}{dt}\gamma^T(t)|_{t=0} = -\gamma^T(t)\frac{d}{dt}\gamma(t)\gamma^{-1}(t)|_{t=0} = -IvI^{-1} = -v$$

and the symmetry at arbitrary element $g \in O(n)$ is given by $s_g = gs_I g^{-1}$ and

$$s_g(x) = gs_I g^{-1}x = gs_I(g^T x) = gx^T g$$

∮.5 Compact Lie groups

let S = G be a compact Lie group with biinvariant Riemannian metric, i.e. left and right translations L_g, R_g are isometries for any $g \in G$. besides, consider the following map

$$s_e(g) = g^{-1}$$

since $s_e(e) = e$ and $ds_e|_e(v) = \frac{d}{dt}\gamma^{-1}(t)|_{t=0} = -\gamma^{-1}(t)\frac{d}{dt}\gamma(t)\gamma^{-1}(t)|_{t=0} = -eve = -v$, if s_e is also a isometry, then it's isotropy at the point e and by theorem 1, it's a symmetry space.

since $ds_e|_e = -I$ which preserve the length of the tangent vector of T_eG, and we know for any $g \in G$

$$s_e L_g = R_{g^{-1}} s_e$$

which means that

$$ds_e|_g \circ dL_g|_e = dR_{g^{-1}}|_e \circ ds_e|_e \rightarrow ds_e|_g = dR_{g^{-1}}|_e \circ ds_e|_e \circ \circ dL_g|_e^{-1}$$

which show that $ds_e|_g$ preserve the length of the vectors in T_gG since L_g and R_g are isometry and $ds_e|_e = -I$ which preserve the length of the tangent vector of T_eG. this shows that s_e is an isometry.

so G is a symmetry space and we have

$$G =< s_e, L_g, R_g > / < s_e, L_g \circ R_{g^{-1}} >$$

∮.6 Projection model of the Grassmannians

let $S = G_k(R^n)$ be the set of the all k dimensional linear subspaces of R^n,which is called Grassman manifold. then the isometry group is $G = O(n)$. and the isotropy group of the standard k dimensional subspace is $K = O(k) \times O(n-k)$, so we have

$$S = O(n)/(O(k) \times O(n-k))$$

the symmetry s_E at a specific point E is the reflection along this subspace E, namely, if F can be decomposed as $F_E \oplus F_{E^\perp}$, then

$$s_E(F) = F_E \oplus -F_{E^\perp}$$

in the following, we try to consider the Riemann Metric structure on this symmetry space.

we try to consider the following map

$$S \to Hom(R^n, R^n) \;\; E \to P_E$$

with P_E be the projection to the E subspace. which is equivariant in the sense that

$$gP_E g^T = P_{gE}$$

for any $g \in O(n)$

since $Hom(R^n, R^n) \cong M(n)$ and P_E is symmetric, so P_E contained in the space $S(n) \subset M(n)$ which is the space of symmetric metrics. furthermore $P_E^2 = P_E$, so it also contained in the space

$$P(n) := \{p \in S(n); p^2 = p\}$$

P_E is k dimensional subspace projection, $\text{tr}(P_E) = k$, so the Grassmann manifold can be identified as the following through the map $E \to P_E$

$$P(n)_k = P(n) \cap S(n)_k \qquad S(n)_k := \{x \in S(n); \text{tr}(x) = k\}$$

then $P(n)_k$ is the orbit of the following standard subspace projection P_{E_0} under the conjugate action of the group O(n) on S(n):

$$P_{E_0} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

and the isotropy group of P_{E_0} is $O(k) \times O(n-k) \subset O(n)$. we consider the corresponding Lie algebra, since $T_I O(n) - T_I(O(k) \times O(n-k))$ has the following form

$$\begin{pmatrix} 0 & -L^T \\ L & 0 \end{pmatrix}$$

for arbitary $L \in R^{(n-k)\times k}$, so the dimension of the grassmann manifold is k(n-k).

besides consider the map on S(n) defined by

$$S(n) \to S(n) \quad\; p \to F(p) = p^2 - p$$

since $G_k(R^n) \subset \ker(F)$, so we know that $T_{p_E} G_k(R^n) \subset \ker(dF_{p_E})$, by the way

$$\ker(dF_{p_E}) = \{v \in S(n); vp_E + p_E v = v\}$$

since $1 - p_E = p_{E^\perp}$, the above equation means that $vp_E = p_{E^\perp} v$, so for any vector $u \in R^n$, we can decompose it as $u = u_E + u_{E^\perp}$, since $v(u_E) = vp_E(u_E) = p_{E^\perp} v(u_E)$, so we have $v(u_E) \in E^\perp$. similarly, $v(u_{E^\perp}) = p_{E^\perp} v(u_{E^\perp}) = vp_E(u_{E^\perp}) = 0$, this means that v is a linear map from E to $E^\perp$, which means that

$$\ker(dF_{p_E}) = \text{Hom}(E, E^\perp)$$

so $\dim(\ker(dF_{p_E})) = k(n-k)$, which means that

$$T_{p_E} G_k(R^n) = \ker(dF_{p_E})$$

then we can use the metrics on $S(n)$ to induce the Riemann metrics in the Grassmainn Manifold $G_k(R^n)$

so ,finally we have

$$G_k(R^n) \to S(n)_k$$
$$E \to P_E$$
$$gE \to gP_E g^T = P_{gE}$$

suppose s_E is the reflection along E in the space $G_k(R^n)$(the symmetry at E), so we can convert it to the language that in space $S(n)_k$ with $\hat{s}_E$ defined by

$$\hat{s}_E(x) = s_E x s_E^T = s_E x s_E$$

we show that $\hat{s}_E$ is a symmetry at P_E,firstly we have $\hat{s}_E(P_E) = s_E P_E s_E = P_E$, secondly $d\hat{s}_E|_{p_E}(v) = s_E v s_E$, we have

$$\begin{aligned} d\hat{s}_E|_{p_E}(v)(u) &= \hat{s}_E(v)(u_E + u_{E^\perp}) = s_E v s_E(u_E) + s_E v s_E(u_{E^\perp}) \\ &= s_E v(u_E) + s_E v(-u_{E^\perp}) = -v(u_E) + 0 = -v(u_E + u_{E^\perp}) \\ &= -v(u) \end{aligned}$$

since $v \in \mathrm{Hom}(E, E^\perp)$ if v belongs to $T_{p_E}S(n)_k$, then the above equation means that $d\hat{s}_E|_{p_E}$ is just -I as expected. so $\hat{s}_E$ is the symmetry at p_E

∮.7 Reflection model of the Grassmannians

similar to the projection model of the grassmainn manifold, we can consider the reflection representation of the linear subspace, suppose E is a k dimensional linear subspace of R^n, we can consider the map of the following

$$G_k(R^n) \to \mathrm{Hom}(R^n, R^n)$$
$$E \to s_E$$

where s_E is the reflection along the E space, which reflect the vector component in the complement of E. since $s_E + I = 2p_E$, with p_E the projection model. so we can identify this Grassmannians as the following space $R(n)_k$

$$R(n) = O(n) \cap S(n) \qquad R(n)_k = \{s \in R(n); \mathrm{tr}(s) = 2k - n\}$$

∮.8 Complex Structures on R^n

Let S be the set of orthogonal complex structures in R^n for even n = 2m. The elements of S are real orthogonal n × n-matrices j with $j^2 = -$I. and we also have $j = -j^T$, thus we have

$$S = O(n) \cap A(n) = \{j \in A(n); j^2 = -1\}$$

since the eigen value of j are all $\pm i$ with multiplicity m, so all S is in a conjugate class, an orbit under the action of O(n) by conjugation,namely, reduced from the $< s_I, L_g, R_{g'} >$(the isometry group of O(n)) to $< L_g R_{g^{-1}} >= O(n)$, so the isometry group of the S is just the $G = O(2m)$. if we consider a standard complex structure denoted as

$$j_0 = I_m \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

then the isotropy group of this complex structure i is the metrics $g \in O(n)$ satisfying

$$gj_0g^{-1} = j_0$$

if we write g as $A \otimes B$, this constrain is just that $B\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}B$, which means that B has the form of $aI + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, since then $g = A \otimes (aI + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$, if we regard the latter one as the complex number due to $j_0^2 = -1$, namely $aI + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = a + bi = c$, then the constrain that $gg^T = I$, implies that $(A \otimes c)(A^T \otimes (a - bi)) = A_cA_c^\dagger = I$, which means that g is an unitary metrics, since $c^T = a - bi = c^*$ and $A \otimes c$ is just the complexity of the real metrics A. so the isotropy group of j_0 is just $K = U(m)$, then we have:

$$S = O(2m)/U(m)$$

then we consider the tangent space, consider the defining map of S from $O(n)$ to $O(n)$, namely

$$F : j \to F(j) = j^2 + 1$$

then $S = \ker(F_j)$ and we have that

$$T_jS = \ker(dF_j) = \{v \in A(n); vj + jv = 0\}$$

which consists of the metrics anticommuting with j. as for the space $T_jU(m)$, since $gjg^{-1} = j$, so

$$T_jU(m) = \{v \in M(n); v^T = -jvj, vj = jv\}$$

and the tangent space of O(2m) at the point j is just

$$T_jO(2m) = \{v \in M(n); v^T = -jvj\}$$

since $j^T = -j$, so we also have if $v \in T_jO(2m)$, then $v = -jv^Tj$, thus we have

$$v + v^T = -j(v + v^T)j \to j(v + v^T) = (v + v^T)j$$
$$v - v^T = -j(v^T - v)j \to j(v - v^T) = -(v - v^T)j$$

since $v = \frac{1}{2}(v + v^T) + \frac{1}{2}(v - v^T)$, so the element of $T_jO(2m)$ can be decomposed of the part which is commute with j and the part which is anti-commute with j, and this two part correspond to the space $T_jU(m)$ and T_jS, since only the commute part consists a closed sub lie-algebra, and the symmetry space is the anti commute part, which is not closed under lie bracket, so we use the quotient to derive this symmetry space in light of the basic groups

$$S = O(2m)/U(m)$$
$$T_j(O(2m)) = T_j(U(m)) \oplus T_j(S)$$

$\oint$.9 Real structures on on C^n

Let S be the set of real structures on C^n. A real structure on $C^n = R^{2n}$ is a reflection κ at a totally real subspace E of half dimension where "totally real"means $iE \perp E$. In other words, κ is a reflection which is complex antil-inear.

since κ is symmetric, if we define $S(2n)_-$ as the intersection of S(2n) with the space of complex antilinear maps on C^n. since complex antilinear maps in C^n can be regard as the reflection with respect to the real part E if we view C^n as the $E \oplus iE = R^{2n}$.

since reflection can be viewed as the subspace of symmetry and orthogonal operators, if further require the anti-linear ,then it means the -1 eigen-value exists, so only reflection remains. if we denote the complex structure of C^n as j,then we have

$$S = S(2n)_- \cap O(2n) = \{\kappa \in R(2n)_n; \kappa j = -j\kappa\}$$

then we try to figure out the isometry group. since $U(n) \subset O(2n)$ is the element g which is commute with j, so we have

$$jg\kappa g^{-1}j = gj\kappa jg^{-1} = g\kappa g^{-1} \quad (g\kappa g^{-1})^2 = g\kappa^2 g^{-1} = gg^{-1} = I$$

which means that U(n) is the isometry group of S.

consider the standard real structure of complex conjugate in C^n, which is anti-linear and refection over E, E is the real span of a unitary basis of C^n. namely $\kappa_0(v) = \bar{v}$. the isotropy group of this element is satisfy that $g\kappa_0 = \kappa_0 g$, which means that g is real metrics, together with the fact that g is element of U(n), so g should be O(n). so the isotropy group is O(n), then

$$S = U(n)/O(n)$$

then we try to figure out the tangent space, for the map $F(x) = x^T x - I$ define $S \subset S(2n)_-$, we have

$$\ker(dF_\kappa) = \{v \in S(2n)_-; v^T\kappa + \kappa^T v = v\kappa + \kappa v = 0\}$$

thus $v \in \ker(dF_\kappa)$ iff the C-linear map κv is a real anti-symmetric $(\kappa v)^T = v\kappa = -\kappa v$, thus $\kappa v \in T_I U(n)$

Moreover, κv anticommutes with κ, so it is purely imaginary with respect to the real structure κ.On the other hand, the purely imaginary matrices in $T_I(U(n))$ form a complement to $T_I(O(n))$, so

$$\ker(dF_\kappa) = T_\kappa(S) \qquad T_I(U(n)) = T_I(O(n)) \oplus T_\kappa S$$

and the symmetry s_κ is given by the conjugation with κ, i.e. $s_\kappa(x) = \kappa x\kappa$. it fix κ and act as -I in $T_\kappa(S)$ since $\kappa v\kappa = \kappa(-\kappa v) = -v$.

$\Re$.3 Transvections and Holonomy

Let γ be the geodesic segment connecting p and q such that $\gamma(0) = m$ is the mid point, and extend it to a complete geodesic. suppose X(s) is a vector field along γ, namely

$$\frac{d}{dt}\gamma(t) = X(t)$$

then consider the vector induced vector field by the symmetry at m, $\tilde{\gamma}(t) = s_m(\gamma(t)) = \gamma(-t)$, then

$$\tilde{X}(t) = \frac{d}{dt}\tilde{\gamma}(t) = \frac{d}{dt}\gamma(-t) = -\frac{d}{d(-t)}\gamma(-t) = -X(-t)$$

this means that the symmetry in m s_m induced a vector fields mapping

$$ds_m(X(t)) = -X(-t)$$

then if we consider the composition of the symmetry at two different points, $\tau = s_q \circ s_m$, then this isometry will induced an vector field mapping following (if $q = \gamma(\frac{l}{2})$)

$$\tau(\gamma(t)) = \gamma(t+l) \qquad d\tau|_{\gamma(t)}.X(t) = X(t+l) \tag{18}$$

which means that this kind of isometry consisting of the composition of two symmetry is a parallel translation along the geodesic, such kind of isometry is called transvection along γ. this kind of transvections form a one-parameter subgroup of the isometry group G. so we have the following theorem:

Theorem §1.1 Each complete geodesic γ of the symmetry space, is the orbit of a one-parameter group of isometries, the transvections along γ, which act as parallel transports along γ.

$\Re$.4 Killing Fields

Let S be a symmetric space and fix a base point $p \in S$. Let $\mathfrak{g}$ be the Lie algebra of the isometry group G of S viewed as the space of Killing vector fields. namely choosing a curve in G as $g(t)$ with $g(0) = I$, then the isometry acting on the base point forms a curve in S, $\gamma(t) = g(t)(p)$, this curve induced an vector field over S, which must be a killing filed X(t) due to g(t) preserve the length in S, all these vector filed X(t) forms the Killing filed representation of the Lie algebra for the isometry group G.

$$X_p = \frac{d}{dt}(\gamma(t)(p))|_{t=0}$$

as for the isotropy group at p, this induced Killing filed is zero at p since if $\gamma(t) \in K$, then $\gamma(t)(p) = p$ which is independent of t, so

$$X_p = \frac{d}{dt}(p)|_{t=0} = 0$$

so those Killing vectors satisfying $X_p = 0$ forms the lie algebra $\mathfrak{h} := \{X \in \mathfrak{g}; X|_p = 0\}$ of the isotropy group at p, $K = G_p$.

we claim that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$

where $\mathfrak{p} := \{X \in \mathfrak{g}; (\nabla X)|_p = 0\}$, which is the space of infinitesimal transvections at p!

since for any geodesic at p with velocity $v \in T_pS$, there is one parameter group of transvections for this geodesic, labeled as $g_v(t) \in G$, then this one parameter group induced a vector filed over S by the curve $\gamma_v(t) = g_v(t)(p)$, so the Killing filed of this parameter group is

$$X_p = \frac{d}{dt}(\gamma_v(t)(p))|_{t=0}$$

we have the derivative of V with respect to $\omega \in T_pS$ is the infinite small parallel transport of the vector field X along the curve defining ω, namely, if p(s) is a curve in S start at p with velocity ω, then

$$\nabla_\omega X|_p = \frac{d}{ds}\frac{d}{dt}g_v(t)(p(s))|_{t=0,s=0} = \frac{d}{dt}\frac{d}{ds}g_v(t)(p(s))|_{t=s=0} = \frac{d}{dt}(dg_v(t).\omega)|_{t=0} = 0$$

since $dg_v(t).\omega$ is the parallel transport of ω. this means that infinitesimal transvections at p are in $\mathfrak{p}$. since a Killing field is determined by its value and first derivative at a single point. so $\mathfrak{h}$ and $\mathfrak{p}$ consists all of $\mathfrak{g}$, since the first one has a degree of freedom on $(\nabla X)_p$ and the latter one has a degree of freedom on X_p which cover all the cases and the dimesion also matches.

moreover, we have the following theorem about these algebra struture

Theorem §1.2 Let S be a symmetric space and $p \in S$, let $\mathfrak{h}$ be the set of Killing fields vanishing at p and let $\mathfrak{p}$ the set of infinitesimal transvections at p, i.e. the Killing fields with vanishing covariant derivative at p. Then

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h} \quad [\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h} \tag{19}$$

further, the map $\mathfrak{p} \to T_pS, V \to V_p$ is a linear isomorphism, and for all U,V,W in $\mathfrak{p}$, we have

$$(R(U,V)W)|_p = ([U,[V,W]])|_p$$

where R is the Ricci tensor.

we can see this since if X,Y in $\mathfrak{h}$, since $X_p = 0$, we have $\nabla_X Y = 0$, similarly, since $Y_p = 0$, we have $\nabla_Y X = 0$, so we have $[X,Y] = \nabla_X Y - \nabla_Y X = 0$ at p, this prove that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$.

similarly, if V,W in $\mathfrak{p}$, then $\nabla_V W = 0$ since $\nabla W = 0$,similarly, $\nabla_W V = 0$ since $\nabla V = 0$ at p.so we have $[V,W] = \nabla_V W - \nabla_W V = 0$ at p this prove $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}$.

so we only need to consider $[X,W] = \nabla_X W - \nabla_W X$, since $[X,W]|_p = -(\nabla_W X)_p$ which can be non zero. so we only need to prove that $\nabla_U[X,W]$ vanishing at p. this can be achieved by using the Bianchi identity and the properties of the Killing Field, we omit this tedious proof here.

ℜ.5 Cartan Involution and Cartan Decomposition

Theorem §1.3

a)Let G be a connected Lie group with an involution (order-2 automorphims) $\sigma : G \to G$ and a left invariant metric which is also right invariant under the closed subgroup

$$\hat{K} = \mathrm{Fix}(\sigma) = \{g \in G; g^\sigma = g\}$$

Let K be a closed subgroup of G with

$$\hat{K}^\circ \subset K \subset \hat{K}$$

where $\hat{K}^\circ$ denote the connected component (identity component) of $\hat{K}$. then $S = G/K$ is a symmetric space where the metric is induced from the given metric on G.

b)Every symmetric space S arises in this way.

comments: we get a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{h}$ and $\mathfrak{p}$ are the eigenspaces of σ_* corresponding to the eigenvalues 1 and -1

Theorem §1.4 A vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ of a Lie algebra $\mathfrak{g}$ is the eigenspace decomposition of an order-two automorphism σ_* of $\mathfrak{g}$ if and only if

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h} \quad [\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h} \tag{20}$$

comments: this is simply the fact that

$$\sigma_*([E_\lambda, E_\mu]) = [\sigma_* E_\lambda, \sigma_* E_\mu] = [\lambda E_\lambda, \mu E_\mu] = \lambda\mu[E_\lambda, E_\mu]$$

References

[1] M. Stone, C.-K. Chiu, and A. Roy, *Symmetries, dimensions and topological insulators: the mechanism behind the face of the bott clock*, Journal of Physics A: Mathematical and Theoretical **44**, 045001 (2010).

[2] A. Altland and M. R. Zirnbauer, *Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures*, Phys. Rev. B **55**, 1142 (1997).