QuantumFieldTheory

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contents

Υ.1 The Klein-Golden Field

ℜ.1 classical point of view

for a real klein-Golden field the quantaty is listed below: lagrangian density:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

motion equation

$$(\partial^\mu\partial_\mu + m^2)\phi = 0$$

Hamitonian density:

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

using fourier transformation:

$$\phi(\vec{x},t) = \int\frac{d^3\vec{p}}{(2\pi)^3}e^{i\vec{p}\bullet\vec{x}}\phi(\vec{p},t)$$

the motion eqaution become:

$$\frac{\partial^2}{\partial t^2} + (|p|^2 + m^2) = 0$$

let $w_{\vec{p}}^2 = |\vec{p}|^2 + m^2$ one can get the motion eauation just like ocsillator:

$$\frac{\partial^2}{\partial t^2} + w_p^2 = 0$$

ℜ.2 quantilization of K-G Field

$$[\phi(x),\pi(y)] = i\delta^{(3)}(x-y)$$

$$\phi(x) = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p e^{ipx} + a_p^\dagger e^{-ipx}) = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p + a_{-p}^\dagger)e^{ipx}$$

$$\pi(x) = \int\frac{d^3\vec{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}(a_p e^{ipx} - a_p^\dagger e^{-ipx}) = \int\frac{d^3\vec{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}(a_p - a_{-p}^\dagger)e^{ipx}$$

$$[a_p, a_{p'}^\dagger] = (2\pi)^3\delta^{(3)}(p-p')$$

to integrate the hamitonian densenty, we get the hamitonnian:

$$H = \int\mathcal{H}d^3x = \int\frac{d^3\vec{p}}{(2\pi)^3}w_p(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger])$$

to make a similar calculation, we get the monnmentum of the field:

$$P = -\int d^3x\pi(x)\nabla\phi(x) = \int\frac{d^3\vec{p}}{(2\pi)^3}pa_p^\dagger a_p$$

the state is defined as:

$$|p\rangle = \sqrt{2E_p}a_p^\dagger|0\rangle$$

and the interpratation of $\phi(x)|0\rangle$ is that create a partical at position x. and there are some realtions:

$$[H, a_p] = -a_p E_p$$

$$[H, a_p^\dagger] = a_p^\dagger E_p$$
$$\phi(x,t) = e^{iHt}\phi(x)e^{-iHt} = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p e^{-ipx} + a_p^\dagger e^{ipx})|_{p_0=E_p}$$
$$\pi(x,t) = \frac{\partial}{\partial t}\phi(x,t)$$

we shold notice that the inner producr in the above relation is lorentz four vector's inner product.

∮.1 K-G Propagator

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$
$$[\phi(x),\phi(y)] = D(x-y) - D(y-x)$$

the retared Green's Function:

$$D_R(x-y) = \theta(x_0-y_0)\langle 0|[\phi(x),\phi(y)]|0\rangle = \theta(x_0-y_0)\int \frac{d^3\vec{p}}{(2\pi)^3}\int \frac{dp_0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip(x-y)}$$

there we introduce a fomilism for delta function:

$$(\partial_t\delta(t))f(t) = -(\partial_t f(t))\delta(t)$$

the motion eauqtion of the retarded green function is :

$$(\partial^2 + m^2)D_R(x-y) = -i\delta^{(4)}(x-y)$$

the fourier tranfer of the green function is:

$$D_R(p) = \frac{i}{p^2-m^2}, D_R(x-y) = \int \frac{d^4p}{(2\pi)^4)}\frac{i}{p^2-m^2}e^{-ip(x-y)}$$

the feymann progatator for a klein-golden partial:

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4)}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}$$

ϒ.2 The Dirac Field.

the lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$

ℜ.1 representation of the lorentz group espically for 4 dimensions

if we define

$$J^{\mu,\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

then the six operator generate the three boost and three rotation of the lorentz group.

$$[J^{\mu,\nu}, J^{\rho,\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

to clearly see theis operators is the generator ,we can use them to form the lorentz transfer:

$$V \to V' = (I - \frac{iJ^{\mu\nu}}{2}w_{\mu,\nu})V$$

in the above description, the $w_{\mu,\nu}$is just elements a random metric w which describ a lorentz tranferomation.

dirac's trick for the representation of lorentz group for n dimension:

if γ^μ is the n dimension metrics satisfying the relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$$

then the six metrics:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

is the generator of the lorentz group for n dimensional representation(to prove this, we just need to show the commutation relations).

ℜ.2 the dirac algebra

the dirac γ metrics:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2}$$

then use the dirac's trick and we get the generator of the lorentz group:

$$S^{0i} = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} S^{ij} = \frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = \frac{1}{2}\epsilon^{ijk}\Sigma^k \tag{3}$$

some properties of the generator:

$$[\gamma^\mu, S^{\rho,\sigma}] = (J^{\rho\sigma})^\mu_\nu \gamma^\nu$$

$$\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu_\nu \gamma^\nu$$

$$\Lambda_{\frac{1}{2}} = exp(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu})$$

since the metrics $S^{\mu\nu}$ is not hermitian, so we should take care of it when related to corresponding calculations.

some properties of these metrics:

$$\sigma^2 \vec{\sigma}^* = -\vec{\sigma}\sigma^2$$

$$(p \bullet \sigma)(p \bullet \bar{\sigma}) = p^2$$

we define 4 vector :

$$\sigma^\mu = (1, \vec{\sigma}), \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

then the gamma metrics have a unit form:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{4}$$

we can use sixteen constant metrics to form a basis for the 4-dimensional metrics space:

$$1, \gamma^\mu, \gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]}, \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}, \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}$$

and we can use $\gamma^5$to simply the expresion for the last 5 metrics:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

we can the clrearly see that:

$$\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5$$

$$\gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = -i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$$

the properties of the $gamma^5$:

$$(\gamma^5)^\dagger = \gamma^5$$

$$(\gamma^5)^2 = 1$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

$$[\gamma^5, S^{\mu\nu}] = 0$$

in dirac's representaion, we have:

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

the standard choice of theses metrics:

$$1, \gamma^\mu, \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \gamma^\mu\gamma^5, \gamma^5$$

a property of the Pauli metrics:

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$$

$$(\bar{\sigma}^\mu)_{\alpha\beta}(\bar{\sigma}_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$$

ℜ.3 classic solution to dirac equation

the weyl spinor:

$$i\bar{\sigma}\partial\Psi_L = 0$$

$$i\sigma\partial\Psi_R = 0$$

the solution to the dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0$$

using fourier tranfer we get the solotion for positive frequency:

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4)} u(p)e^{-ipx} \rightarrow (p_\mu \gamma^\mu - m)u(p) = 0$$

the solution is :

$$u^s(p) = \begin{pmatrix} \sqrt{p \bullet \sigma}\xi^s \\ \\ \sqrt{p \bullet \bar{\sigma}}\xi^s \end{pmatrix} \tag{6}$$

and the normalization is :

$$\bar{u}^r u^s = 2m\delta^{r,s} \rightarrow (u^r)^\dagger u^s = 2E_p \delta^{r,s}$$

the helicity operator:

$$\hat{h} = \hat{p} \bullet S = \frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ \\ 0 & \sigma^i \end{pmatrix}$$

using fourier tranfer we get the solotion for negetive frequency:

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4)} v(p)e^{ipx} \rightarrow (p_\mu \gamma^\mu + m)v(p) = 0$$

the solution is :

$$v^s(p) = \begin{pmatrix} \sqrt{p \bullet \sigma}\eta^s \\ \\ -\sqrt{p \bullet \bar{\sigma}}\eta^s \end{pmatrix} \tag{7}$$

and the normalization is :

$$\bar{v}^r v^s = -2m\delta^{r,s} \rightarrow (v^r)^\dagger v^s = 2E_p \delta^{r,s}$$

$$\sum_s \bar{u}^s(p)u^s(p) = \gamma \bullet p + m$$

$$\sum_s \bar{v}^s(p)v^s(p) = \gamma \bullet p - m$$

ℜ.4 quantilization of the dirac field

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

$$H = \int d^3x \bar{\Psi}(-i\gamma \bullet \nabla + m)\Psi = \int d^3x \Psi^\dagger(-i\gamma_0 \gamma \bullet \nabla + m\gamma_0)\Psi$$

the quantilized dirac field is:

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a_p^s u^s(p)e^{-ipx} + (b_p^s)^\dagger v^s(p)e^{ipx})$$

the anticommutation relations are:

$$\{a_p^r, (a_q^s)^\dagger\} = \{b_p^r, (b_q^s)^\dagger\} = (2\pi)^3 \delta^3(p - q)\delta^{rs}$$

$$\{\Psi_a(x), \Psi_b^\dagger(y)\} = \delta^3(x - y)\delta_{ab}$$

then the hamitonian is :

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p((a_p^s)^\dagger a_p^s + (b_p^s)^\dagger b_p^s)$$

the total monmentum operator is:

$$P = \int \frac{d^3p}{(2\pi)^3} \sum_s p((a_p^s)^\dagger a_p^s + (b_p^s)^\dagger b_p^s)$$

the angle monmentum is :

$$J = \int d^3x \Psi^\dagger (\vec{x} \times (-i\nabla) + \frac{1}{2}\Sigma)\Psi$$

the total charge:

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s ((a_p^s)^\dagger a_p^s - (b_p^s)^\dagger b_p^s)$$

∮.1 the feyman propagator for dirac field

the retared green function:

$$S_R(x-y) = (i\partial\!\!\!/_x + m)D_R(x-y)$$

the greenn function satisfy the equation:

$$(i\partial\!\!\!/_x - m)S_R(x-y) = i\delta^4(x-y)I$$

the fourier transform of the retarded green function is:

$$S_R(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2}$$

the feymann propagator:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-i(x-y)}$$

$$S_F(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}$$

ℜ.5 discrete symmetrics in dirac field

parity P,time reversal T,and chage interchage C

1.Parity P:reverse the momentum but preserve the spin:

$$a_p^{s^\dagger}|0\rangle \xrightarrow{P} a_{-p}^{s^\dagger}|0\rangle$$

$$Pa_p^s P = \eta_a a_{-p}^s$$

$$Pb_p^s P = \eta_b b_{-p}^s$$

$$\eta_a \eta_b = -1$$

$$P\Psi(t,x)P = \eta_a \gamma^0 \Psi(t,-x)$$

$$P\Psi(\bar{t},x)P = \eta_a^* \Psi(\bar{t},-x)\gamma^0$$

2.Time Reversal T:reverse the momentum and spin time reversal operator also act on the c-number:

$$T(c) = c^* T$$

define two vector operator:

$$a_p^s == (a_p^2, -a_p^1), b_p^s == (b_p^2, -b_p^1)$$

then time reversal operator T has the property:

$$Ta_p^s T = a_{-p}^{-s}, Tb_p^s T = b_{-p}^{-s}$$

$$T\Psi(t,x)T = -\gamma^1\gamma^3\Psi(-t,x)$$

$$T\bar{\Psi}(t,x)T = \bar{\Psi}(-t,x)\gamma^1\gamma^3$$

3.charge conjugation C

$$Ca_p^s C = b_p^s, Cb_p^s C = a_p^s$$

$$C\Psi(x)C = -i(\bar{\Psi}\gamma^0\gamma^2)^T$$

$$C\bar{\Psi}C = (-i\gamma^0\gamma^2\Psi)^T$$

Υ.3 The Vector Field

Since the Largaran of the eletromagnetic field is:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

so we can easily compute the quantity(which is myself defined):

$$\pi^{\mu\nu} = \frac{\partial L}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu}$$

which implies the conjugate monmentum to A_0 is:

$$\pi^0 = \frac{\partial L}{\partial(\partial_0 A_0} = F^{00} = 0$$

so the standard quantilization is not working here

to make a trick, we can use a new largrange:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\xi}{2}(\partial^\mu A_\mu)^2$$

which is the same as before in lorentz gauge:$\partial^\mu A_\mu = 0$.

in such a form, the quantity as before defined is become:

$$\pi^{\mu\nu} = \frac{\partial L}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu} - \xi\eta^{\mu\nu}(\partial^\sigma A_\sigma)$$

so at this time we have conjugate monmentum about the zero component:

$$\pi^0 = \frac{\partial L}{\partial(\partial_0 A_0}) = -\xi\partial^\sigma A_\sigma = -\xi\partial \bullet A$$

from this formula ,it is clearly to see that if we want to quantelize the eletromagnetic field from the standard procedure ,the lorentz gauge is not working.

and the motion equation is :

$$\partial^2 A_\mu - (1-\xi)\partial_\mu(\partial \bullet A) = 0$$

so when we choose $\xi = 1$ which called the Feymann Gauge,the motion equation is just the same as the classic wavefunction:

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\lambda=0}^{3}(a_p^{(\lambda)}\epsilon_\mu^{(\lambda)}(p)e^{-ipx} + a_p^{(\lambda)\dagger}\epsilon_\mu^{(\lambda)*}(p)e^{ipx})$$

where symbol λ denote the polarization of the photon.

ℜ.1 convention for the polarization vector

first we have p fixed,and we randomly choose a unit vector n which satisfy $n^0 > 0$.at this time we have two vectors fixed.we choose vector $\epsilon^{(1),(2)}$ that in the plane which is vertical to the n and p and satisfying:

$$\epsilon^\lambda(p) \bullet \epsilon^{\lambda'*}(p) = -\delta_{\lambda,\lambda'}, with \lambda, \lambda' = 1,2$$

we choose $\epsilon^3(p)$ in the plane which n and p located and make it vertical to n and is unit:

$$\epsilon^3(p) \bullet n = 0, (\epsilon^3(p))^2 = -1$$

to sommerize we have the relations in our convention:

$$\sum_\lambda \frac{\epsilon_\mu^\lambda(p)\epsilon_\nu^{\lambda*}(p)}{\epsilon^\lambda(p) \bullet \epsilon^{\lambda*}(p)} = \eta^{\mu\nu}$$

$$\epsilon^\lambda(p) \bullet \epsilon^{\lambda'*}(p) = \eta^{\lambda\lambda'}$$

and the quantilization relation is:

$$[A^\mu(x), \dot{A}^\nu] = -i\eta^{\mu\nu}\delta^3(x-y)$$

ϒ.4 Interacting fields and feymann diagrams

a notation about the units:

$$\hbar = c = 1$$

when using natural units,since:

$$E = mc^2 = \hbar\nu = \frac{\hbar}{T} = \frac{\hbar c}{L}$$

so the quantity has these relations:

$$[E] = [M] = [T]^{-1} = [L]^{-1}$$

ℜ.1 Three Interacting System

for the first interacting system,we consider the 4th-phi theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

the second one is the quantum electrodynamics:

$$\mathcal{L}_{QED} = \mathcal{L}_{dirac} + \mathcal{L}_{maxwell} + \mathcal{L}_{in} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2 - e\bar{\Psi}\gamma^\mu\Psi A_\mu$$

and the last one is the Yukawa theory:

$$\mathcal{L}_{Yukawa} = \mathcal{L}_{dirac} + \mathcal{L}_{K-G} - g\bar{\Psi}\Psi\phi$$

ℜ.2 Wick's Theorem

$$T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\cdots\} = N\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\cdots + all - posibble - contractions\}$$

ℜ.3 cross section and S metrics

we have the expression for the cross section in ananolous with S metrics is:

$$d\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}(\prod \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})|M(p_A, p_B \to \{p_f\})|^2(2\pi)^4\delta^{(4)}(p_A + p_B - \sum p_f)$$

and the differential cross section is :

$$(\frac{d\sigma}{d\Omega})_{CM} = \frac{1}{2E_A 2E_B|v_A - v_B|}\frac{|p_1|}{(2\pi)^2 4E_{cm}}|M(p_A, p_B \to \{p_f\})|^2$$

and the decay rate is:

$$d\Gamma = \frac{1}{2m_A}(\prod \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})|M(p_A, p_B \to \{p_f\})|^2(2\pi)^4\delta^{(4)}(p_A + p_B - \sum p_f)$$

where the M is associated with the S metrics which is:

$$\hat{S} = \lim_{t\to\infty} e^{-i2\hat{H}t}$$

$$\hat{S} = 1 + i\hat{T}$$

$$(2\pi)^4\delta^{(4)}(p_A + p_B - \sum p_f)M(p_A, p_B \to \{p_f\}) = \langle p_1, p_2 \cdots |\hat{T}|p_A p_B\rangle$$

for the T operator ,we have the following formula to concuate it:

$$\langle p_1 p_2 \cdots p_n|i\hat{T}|p_A p_B\rangle = lim_{T\to\infty(1-i\epsilon)}({}_0\langle p_1 p_2 \cdots p_n|T(e^{-i\int_{-T}^{T} dt H_I(t)})|p_A p_B\rangle_0)_{connected-and-amputated}$$

Υ.5 Elementary process of quantum electrodynamics

ℜ.1 Some useful relations

$$(\bar{v}\gamma^\mu u)^* = \bar{u}\gamma^\mu v$$

any QED amplitute involving external fermions,when squared or summed over spin or overaged over spins, can be converted to trace of products of dirac metrics.for example for the process of figure 1 we have:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4q^4}trace[(\not{p}' - m_e)\gamma^\mu(\not{p}' + m_e)\gamma^\nu]trace[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu]$$

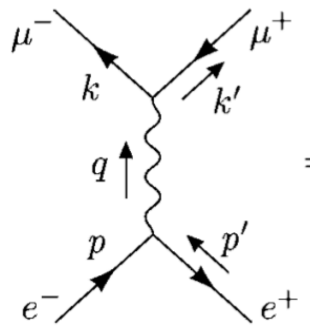
the trace of an odd product of gamma metrics is zero(if n is odd):

$$trace[\gamma_1^\mu \cdots \gamma_n^{\mu'}] = 0$$

$$tr[\gamma^\mu\gamma^\nu] = tr[2g^{\mu\nu}1 - \gamma^\nu\gamma^\mu]$$

for the even number gamma metrics product ,we can anticommute the first one to the right and cycle it back ,we have the trace of two gamma metrics product:

$$tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$$



$$= \bar{v}^{s'}(p')\left(-ie\gamma^{\mu}\right)u^{s}(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^{r}(k)\left(-ie\gamma^{\nu}\right)v^{r'}(k').$$

figure 1: $e^+e^- \to \mu^+\mu^-$process

similarly for the four gamma metrics:

$$tr[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}] = tr[2g^{\mu\nu}\gamma^{\rho}\gamma^{\sigma} - \gamma^{\nu}2g^{\mu\rho}\gamma^{\sigma} + \gamma^{\nu}\gamma\rho 2g^{\mu\sigma} - \gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}]$$

thus we have the following formula:

$$tr[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$,we have the trace fromula related to γ^5:

$$tr[\gamma^5] = 0$$

$$tr[\gamma^{\mu}\gamma^{\nu}\gamma^5] = 0$$

$$tr[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^5] = -4i\epsilon^{\mu\nu\rho\sigma}$$

and there are some formula for the antisymmetric tensor:

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -24$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma'} = -6\delta^{\sigma}_{\sigma'}$$

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho})$$

and there is another useful formula:

$$tr[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\cdots] = tr[\cdots\gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}]$$

if we set $C = \gamma^0\gamma^2$ then we have:

$$C^2 = 1$$

$$C\gamma^{\mu}C = -(\gamma^{\mu})^T$$

when the gamma metrics is dotted inside the trace,one can always simplify it:

$$\gamma^{\mu}\gamma_{\mu} = g_{\mu\nu}\gamma^{\mu}\gamma^{\nu} = \frac{1}{2}g_{\mu\nu}\{\gamma^{\mu},\gamma^{\nu}\} = g_{\mu\nu}g^{\mu\nu} = 4$$

$$\gamma^{\mu}\gamma^{\nu}\gamma_{\mu} = -2\gamma^{\nu}$$

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma_{\mu} = 4g^{\nu\rho}$$

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma_{\mu} = -2\gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}$$

ℜ.2 the unpolarized cross section for the process:$e^+e^- \to \mu^+\mu^-$

when consider that $\frac{m_e}{m_\mu}$ is very small ,we can just set $m_e = 0$,thus:

$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{8e^4}{q^4}[(p\bullet k)(p'\bullet k') + (p\bullet k')(p'\bullet k) + m_\mu^2(p\bullet p')]$$

after a long journey of calculation, we dervie the cross section for this process:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}[1+\frac{m_\mu^2}{E^2} + (1-\frac{m_\mu^2}{E^2})cos^2\theta]$$

and integrate it we can get the total cross section:

$$\sigma_{total} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}(1+\frac{1}{2}\frac{m_\mu^2}{E^2})$$

we can define a unit of R:

$$R = \frac{4\pi\alpha^2}{3E_{cm}^2}$$

which means we regard the cross section for the process $e^+e^- \to \mu^+\mu^-$ as a basic unit.

ℜ.3 $e^+e^- \to \mu^+\mu^-$:helicity structure

$$\frac{d\sigma}{d\Omega}(e_R^-e_L^+ \to \mu_L^-\mu_R^+) = \frac{\alpha^2}{4E_{cm}}(1+cos\theta)^2$$
$$\frac{d\sigma}{d\Omega}(e_R^-e_L^+ \to \mu_R^-\mu_L^+) = \frac{\alpha^2}{4E_{cm}}(1-cos\theta)^2$$
$$\frac{d\sigma}{d\Omega}(e_L^-e_R^+ \to \mu_L^-\mu_R^+) = \frac{\alpha^2}{4E_{cm}}(1-cos\theta)^2$$
$$\frac{d\sigma}{d\Omega}(e_L^-e_R^+ \to \mu_R^-\mu_L^+) = \frac{\alpha^2}{4E_{cm}}(1+cos\theta)^2$$

for a right-hannded spinor ,we have:

$$\hat{p}\cdot\vec{\sigma}\eta = +\eta$$

for a left-handed spinor,we have:

$$\hat{p}\cdot\vec{\sigma}\eta = -\eta$$

some notes onn the bound state:

$$\sigma(e^+e^- \to B) = 4\pi^2\frac{3\Gamma(B\to e^+e^-)}{M}\delta(E_{cm}^2 - M^2)$$

the cross section for e^+e^- to a bound state with spin-1 is that:

$$\sigma(e^+e^- \to B) = 64\pi^3\alpha^2\frac{|\Psi(0)|^2}{M^3}\delta(E_{cm}^2 - M^2)$$

and the decay rate for the bound state is:

$$\Gamma(B\to e^+e^-) = \frac{16\pi\alpha^2}{3}\frac{|\Psi(0)|^2}{M^2}$$

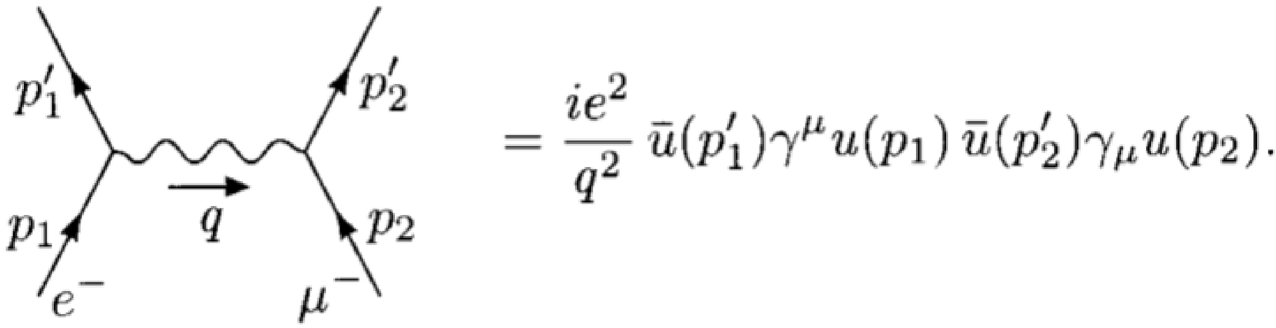


figure 2: $e^-\mu^- \to e^-\mu^-$process

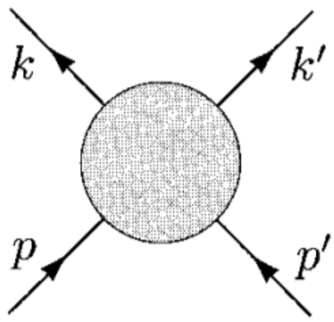


figure 3: illustration for the definition of mandelstam variable which involving 2-body to 2-body scattering process

ℜ.4 $e^-\mu^- \to e^-\mu^-$**scatter**

the process's feymann diagram is presentend in figure 2. just similar to the calculation with $e^+e^- \to \mu^+\mu^-$,we can easily get the deferiential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2k^2(E+k)^2(1-cos\theta)^2}[(E+k)^2 + (E+kcos\theta)^2 - m_\mu^2(1-cos\theta)]$$

Crossing Symmetry:

$$M(\phi(p) + \cdots \to \cdots) = M(\cdots \to \cdots \bar{\phi}(k))$$

where $p\bar{h}i$ is the antipartical of ϕ and k=-p

Mandelstam Variable: when the initial and final state are all two particals we can define the variables called Mandelstam Variable as below(see figure 3 for the process):

$$s = (p+p')^2 = (k+k')^2$$

$$t = (k-p)^2 = (k'-p')^2$$

$$u = (k'-p)^2 = (k-p')^2$$

we can easily work out a relation:

$$s + t + u = \sum_i m_i^2$$

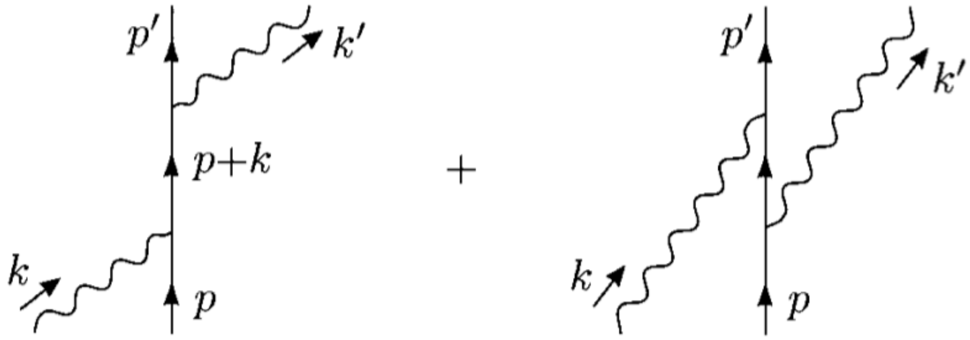


figure 4: compton scattering process

ℜ.5 Compton Scattering

the feymann diagram for this process is presented in figure 4

so the amplitute is :

$$i\mathcal{M} = -ie^2\epsilon_\mu^*(k')\epsilon_\nu(k)\bar{u}(p')[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p \bullet k} + \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p \bullet k'}]u(p)$$

similar to the sum of electron polarization,there is also a good relation for sum of the photon polarization which is:

$$\sum_{polarization} \epsilon_\mu^*\epsilon_\nu \to -g_{\mu\nu}$$

with this in mind, we can easily get the quantity we want:

$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{e^4}{4}\{\frac{I}{(2p\bullet k)^2} + \frac{II}{(2p\bullet k)(2p\bullet k')} + \frac{III}{(2p\bullet k')(2p\bullet k)} + \frac{IIII}{(2p\bullet k')^2}\}$$

after a long journey of calculation we have the relations below:

$$\begin{aligned} I &= tr[(\not{p}' + m)(\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(\not{p} + m)(\gamma_\nu \not{k}\gamma_\mu + 2\gamma_\mu p_\nu)] \\ &= 16(4m^4 - 2m^2 p\bullet p' + 4m^2 p\bullet k - 2m^2 p'\bullet k + 2(p\bullet k)(p'\bullet k)) \\ &= 16(2m^4 + m^2(s-m^2) - \frac{1}{2}(s-m^2)(u-m^2)) \end{aligned} \tag{8}$$

where the s,t,u is the Mandelstam Variables.

similarly we can work out the other three terms:

$$IIII = 16(2m^4 + m^2(u-m^2) - \frac{1}{2}(s-m^2)(u-m^2)) \tag{9}$$

$$II = III = -8(4m^4 + m^2(s-m^2) + m^2(u-m^2)) \tag{10}$$

finally we have :

$$\frac{1}{4}\sum_{spins}|M|^2 = 2e^4[\frac{p\bullet k'}{p\bullet k} + \frac{p\bullet k}{p\bullet k'} + 2m^2(\frac{1}{p\bullet k} - \frac{1}{p\bullet k'}) + m^4(\frac{1}{p\bullet k} - \frac{1}{p\bullet k'})^2]$$

when we work in the fram of lab,we can make all the p,p',k,k' illustrated as figure 5.

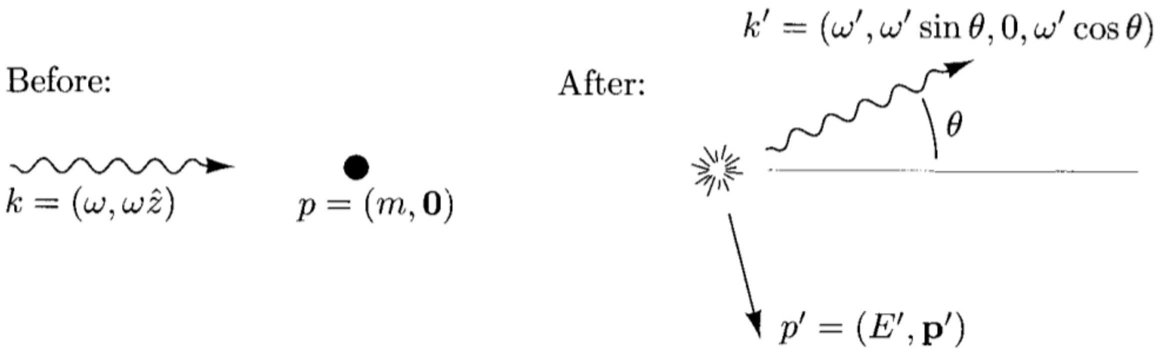


figure 5: compton scattering process in lab reference

thus we have the differtial cross section:

$$\frac{d\sigma}{d(cos\theta)} = \frac{1}{2\omega}\frac{1}{2m}\frac{1}{8\pi}\frac{\omega'^2}{m\omega}(\frac{1}{4}\sum_{spins}|M|^2)$$

$$\frac{d\sigma}{d(cos\theta)} = \frac{\pi\alpha^2}{m^2}(\frac{\omega'}{\omega})^2[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta]$$

where:

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - \cos\theta)}$$

this is called Klein-Nishina-Formula

when $\omega \to 0$,then $\frac{\omega'}{\omega} \to 0$,thus we have:

$$\frac{d\sigma}{d(cos\theta)} = \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta)$$

which is the classical results which is known for all of us.

ϒ.6 Radiative Correlation

Some Identities:

Ward Identity:

$$k_\mu M^\mu = 0$$

Gordon Identity:

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\{\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\}u(p)$$

ℜ.1 Soft Bremsstrahlung

The Current for a arbitory trajectory $y^\mu(\tau)$ is:

$$j^\mu(x) = e\int d\tau\frac{dy^\mu(\tau)}{d\tau}\delta^(4)(x - y(\tau))$$

we can consider a eletron with p get a sudden kick at$(0, \vec{0})$ which illustrated in figure 6 for this process, we can use the current as a source to solve the maxwell equation to get the potential:

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{-ie^2}{k^2}(\frac{p'^\mu}{p \bullet k' + i\epsilon} - \frac{p^\mu}{p \bullet k - i\epsilon})$$

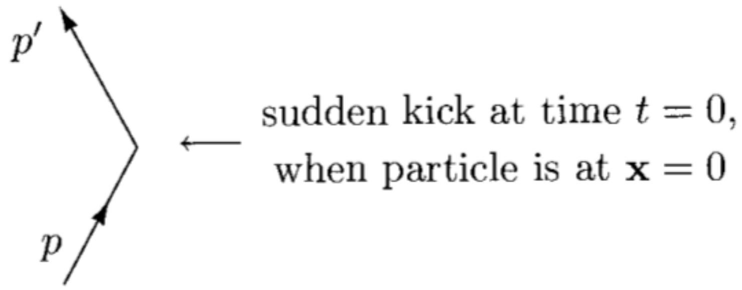


figure 6: Soft Bremsstrahlung case: a electron get a sudden kick.

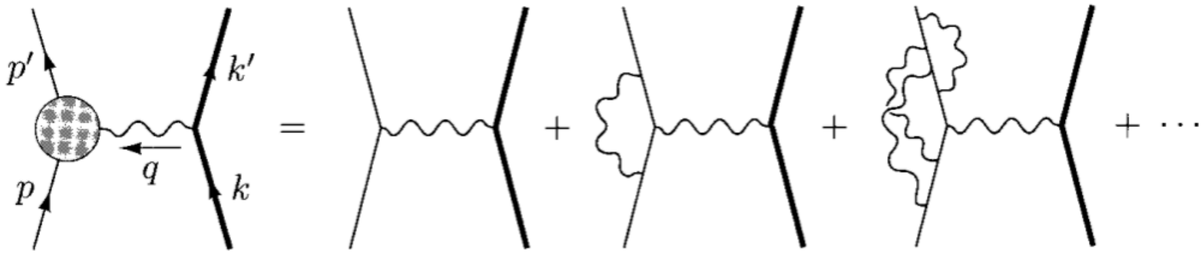


figure 7: Electron Vertex Structure Illustration

and the riadiative part of the potential is:

$$A^{\mu}_{r}ad(x) = Re\int \frac{d^3k}{(2\pi)^3}e^{-ikx}A^{\mu}(k)$$

where:

$$A^{\mu}(k) = \frac{-e}{|\vec{k}|}(\frac{p'^{\mu}}{p \bullet k' + i\epsilon} - \frac{p^{\mu}}{p \bullet k - i\epsilon})$$

and the energy radiatied is:

$$E = \int \frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}\frac{e^2}{2}(\frac{2p \bullet p'}{(k \bullet p')(k \bullet p)} - \frac{m^2}{(k \bullet p')^2} - \frac{m^2}{(k \bullet p)^2})$$

Calculating the same quantity using quantum theory:

$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p')\int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}\sum_{\lambda=1,2} e^2|\epsilon_{\lambda} \bullet (\frac{p'}{p' \bullet k} - \frac{p}{p \bullet k})|^2$$

$$d\sigma(p \to p' + \gamma) =_{-q^2\to\infty} d\sigma(p \to p')\frac{\alpha}{\pi}\ln(\frac{-q^2}{\mu^2})\ln(\frac{-q^2}{m^2})$$

where μ is assumed tiny mass for photon to solve the divergence.

ℜ.2 The Electron Vetex Function

Just as the figure 7 shows,we suppose the vertex is:

$$-ie\Gamma^{\mu}(p', p)$$

Using the Lorentz Invariance and Ward identity we can get the general form for it:

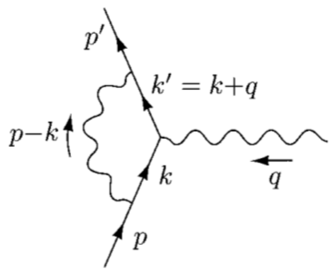


figure 8: the order-α for Γ^μ

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$$

where F_1 and F_2 is unkown function called form factor

for the Lande g-factor we have the ralation:

$$g = 2(F_1(0) + F_2(0))$$

A trick for some integration:

$$\frac{1}{A_1A_2\cdots A_n} = \int_0^1 dx_1\cdots dx_n\delta(\sum x_i - 1)\frac{(n-1)!}{(x_1A_1+\cdots x_nA_n)^n}$$

$$\frac{1}{A_1^{m_1}A_1^{m_2}\cdots A_n^{m_n}} = \int_0^1 dx_1\cdots dx_n\delta(\sum x_i - 1)\frac{\prod x_i^{m_i-1}}{(\sum x_iA_i)^{\sum m_i}}\frac{\Gamma(m_1+\cdots m_n)}{\Gamma(m_1)\Gamma(m_2)\cdots\Gamma(m_n)}$$

Feymann Parameter:

$$\begin{aligned}\frac{1}{(k-p)^2(k^2-m^2)} &= \int_0^1 dxdy\delta(x+y-1)\frac{1}{[x(k-p)^2+y(k^2-m^2)]^2}\\ &= \int_0^1 dxdy\delta(x+y-1)\frac{1}{(k^2-2xk\cdot p+xp^2-ym^2)^2}\end{aligned}$$

when we define $l = k - xp$, then the whole integration is only rely on the l^2 and can trasfer to spherical coordinate to calculate the monmentum integration.the variable x,y is called Feymann Parameter.

we can use Wick rotation to change from Minkovski space to Eculid Space:

$$l^0 = il_E^0; \qquad \vec{l} = \vec{l}_E$$

then the intergration becomes:

$$\int d^4l\frac{1}{(l^2-\Delta)^m} = \frac{i}{(-1)^m}\frac{1}{(2\pi)^4}\int d^4l_E\frac{1}{(l_E^2+\Delta)^m}$$

the right hand side inner product of the above equation is the inner product in Eculid Space. Using all the tricks above ,one can work out the diagram(figure 8):

$$\begin{aligned}&\frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\\ &\times\bar{u}(p')(\gamma^\mu[\ln\frac{z\Lambda^2}{\Delta}+\frac{1}{\Delta}((1-x)(1-y)q^2+(1-4z+z^2)m^2)]\\ &+\frac{i\sigma^{\mu\nu}q_\nu}{2m}[\frac{1}{\Delta}2m^2z(1-z)])u(p)\end{aligned}$$

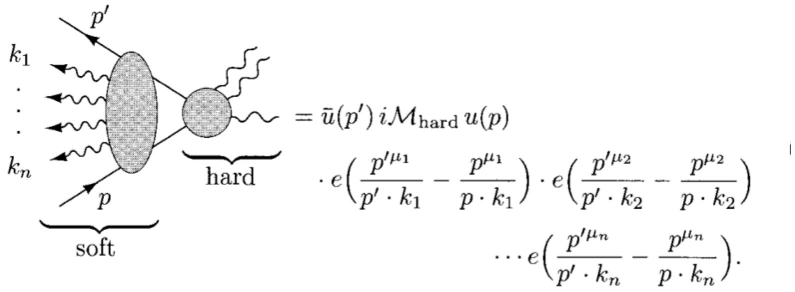


figure 9: One can work out such diagram

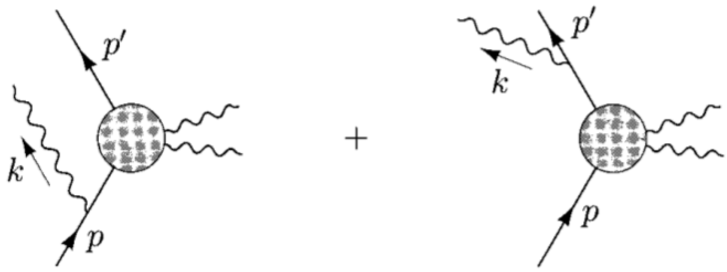


figure 10: scattering a real photon process

where Λ is a Parameter Introduced to solve the divergence.:

$$\frac{1}{(k-p^2)+i\epsilon} \to_{\Lambda\to\infty} \frac{1}{(k-p^2)+i\epsilon} - \frac{1}{(k-p^2)-\Lambda^2+i\epsilon}$$

with such substraction,the divergence of this type:

$$\int dl^2 \frac{l^4}{(l^2+\Delta)^3} \to \int dl^2 (\frac{l^4}{(l^2+\Delta)^3} - \frac{l^4}{(l^2+\Delta_\Lambda)^3})$$

ℜ.3 Summution And Interpreatation of Infrared Divergence.

and for a vitural photon such as the diagram like figure 8,Its value is(when connected to another diagram):

$$\frac{e^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\epsilon}(\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k})(\frac{p'}{-p'\cdot k} - \frac{p}{-p\cdot k}) = X$$

with above calculation :

$$X = -\frac{\alpha}{2\pi} f_{IR}(q^2) \ln\frac{-q^2}{\mu^2}$$

and the differential cross section for scattering a real photon like diagram(figure 10) is:

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}(-g_{\mu\nu})(\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k})(\frac{p'^\nu}{p'\cdot k} - \frac{p^\nu}{p\cdot k}) = Y$$

with above calculation,we have:

$$Y = \frac{\alpha}{\pi} f_{IR}(q^2) \ln\frac{E_l^2}{\mu^2}$$

so we have:

$$\sum_{n=0}^{\infty}\frac{d\sigma}{d\Omega}(p\to p' + n\gamma) = \frac{d\sigma}{d\Omega}(p\to p')e^Y$$

and the correction for arbitrary vitural photon is:

$$\sum_{n=0}^{\infty} \frac{1}{n!} X^n = e^X$$

so the total differential cross section obseverd is:

$$(\frac{d\sigma}{d\Omega})_{measured} = \frac{d\sigma}{d\Omega}(p \to p')e^{2X}e^Y$$
$$= \frac{d\sigma}{d\Omega}(p \to p')e^{-\frac{\alpha}{\pi}f_{IR}(q^2)\ln\frac{-q^2}{E_l^2}}$$

Which Is Not Depend On The Vitural Photon Mass μ

Υ.7 Radiative Correlation:Some formal Development

ℜ.1 Field Strength Renormalization

we can use the spectrol of interacting hamitonian H to create a identity operator,since H is commute with the total momentum P,we can choose the states with:

$$|\Omega\rangle, |\lambda_0\rangle, |\lambda_p\rangle$$

where the relations is:

$$H|\Omega\rangle = 0|\Omega\rangle, P|\Omega\rangle = 0|\Omega\rangle$$

$$H|\lambda_0\rangle = E_0|\lambda_0\rangle = m_\lambda|\lambda_0\rangle, P|\lambda_0\rangle = 0|\lambda_0\rangle$$

$$H|\lambda_p\rangle = E_p|\lambda_p\rangle, P|\lambda_p\rangle = p|\lambda_p\rangle$$

where,

$$E_p = \sqrt{|\vec{p}|^2 + m_\lambda^2}$$

since the state with total momentum zero is not only one,so different λ give the different such states;

then we can use there states to construct the identity operotor:

$$1 = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3)} \frac{1}{2E_\lambda} |\lambda_p\rangle\langle\lambda_p|$$

insert this operotor into the correction function one can get:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4)} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

so we have dervied the Kallen-Lehmann spectral representation of the two-points function:

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int \frac{dM^2}{2\pi} \rho(M^2) D_F(x-y, M^2)$$

and the general form for $\rho(M^2)$ is:

$$\rho(M^2) = \sum_\lambda 2\pi\delta(M^2 - m_\lambda^2)|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

$$\rho(M^2) = 2\pi\delta(M^2 - m_\lambda^2)Z + nothing - untill - M^2 \gtrsim (2m)^2$$

we call Z as the field strength renormalization

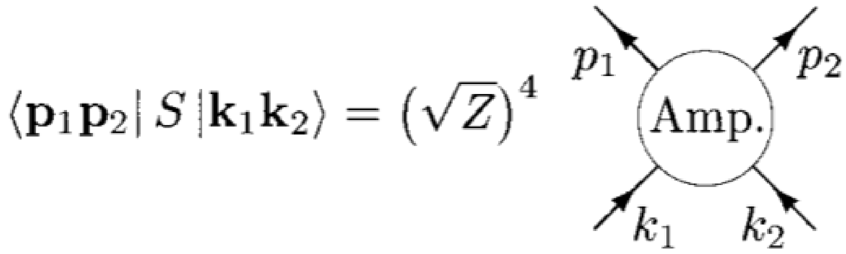


figure 11: Using LSZ formula to calculate the S metrics elements.

and the fourier transfer of the two points function is:

$$\int d^4x e^{ipx} \langle \Omega | T\phi(x)\phi(0) | \Omega \rangle = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^{\infty} \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}$$

and for the dirac firld,the quantity is:

$$\int d^4x e^{ipx} \langle \Omega | T\psi(x)\bar{\psi}(0) | \Omega \rangle = \frac{iZ(\not{p} + m)}{p^2 - m^2 + i\epsilon} + \cdots$$

ℜ.2 The LSZ reduction formula

Lehmann,Symanzik,Zimmerman;

$$\prod_1^n \int d^4x_i e^{ip_i x_i} \prod_1^m \int d^4y_j e^{-ik_j y_j} \langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)\} | \Omega \rangle$$
$$\sim_{p_i^0 \to +E_{\vec{p}_i}, k_j^0 \to +E_{\vec{k}_j}} = (\prod_1^n \frac{i\sqrt{Z}}{p_i^2 - m^2 + i\epsilon})(\prod_1^m \frac{i\sqrt{Z}}{k_j^2 - m^2 + i\epsilon}) \langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle$$

using LSZ reduction formula,one can work out the feymann diagrammatic method for the S metrics elements 11

ℜ.3 The Optical Theorem

since

$$S^\dagger S = 1$$

on can get the result:

$$-i[T - T^\dagger] = T^\dagger T$$

insert a complete set of intermedia states one can get:

$$-i[M(k_1k_2 \to p_1p_2) - M^*(p_1p_2 \to k_1k_2)]$$
$$= \sum_n (\prod_{i=1}^n \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{2E_i}) M^*(p_1p_2 \to \{q_i\}) M(k_1k_2 \to \{q_i\}) \times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_i q_i)$$

so one can get the optical theorem:

$$2ImM(k_1k_2 \to k_1k_2) = 2E_{cm}p_{cm}\sigma_{tot}(k_1k_2 \to anything)$$

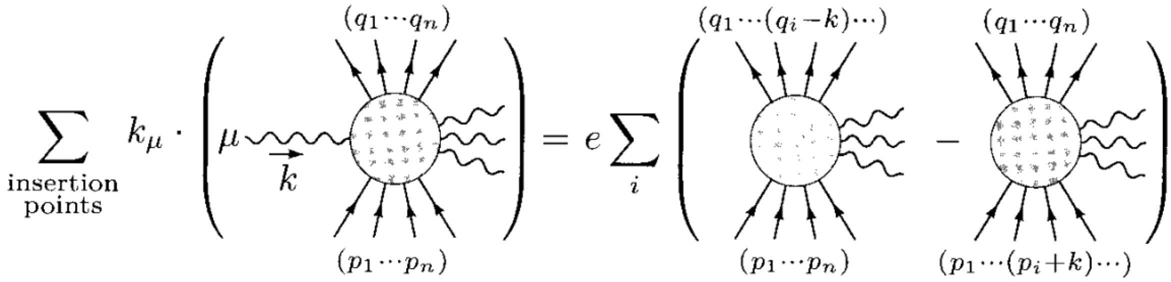
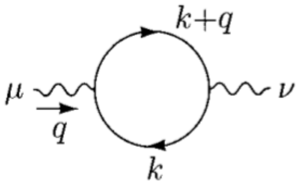


figure 12: The Ward-Takahashi Identity



$$= (-ie)^2(-1) \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left[\gamma^\mu \frac{i}{\not{k} - m} \gamma^\nu \frac{i}{\not{k} + \not{q} - m}\right]$$

$$\equiv i\Pi_2^{\mu\nu}(q). \tag{7.71}$$

figure 13: The feymann diagram for loop

ℜ.4 The Ward-Takahashi Identity

the warden identity is illustated in figure 12.

Ward identity is the diagrammatic expression of the current conservation,which is in turn a consequence of gauge invariance

ℜ.5 Renormalization of the Electric Charge

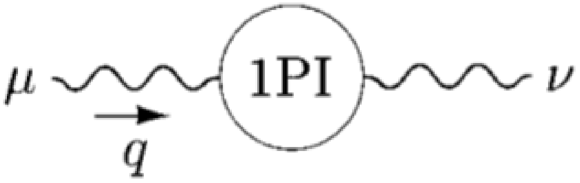
the definition and diagram is illustrated as figure **??** and **??**。from the ward identity ,one can expect the form:

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$$

thus the calculation result is as the figure 15 shows. where the Z_3 is called charge regulization:

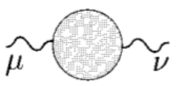
$$Z_3 = \frac{1}{1 - \Pi(0)}$$

$$e_0 \to \sqrt{Z_3} e_0$$



$$\equiv i\Pi^{\mu\nu}(q),$$

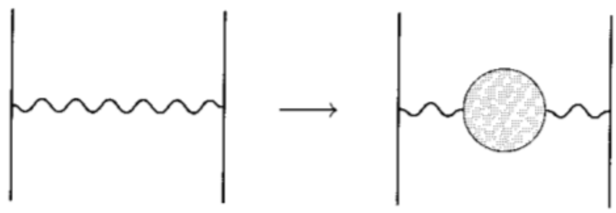
figure 14: The 1PI Part



$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left(\delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2}\right)\left(\Pi(q^2) + \Pi^2(q^2) + \right.$$

$$= \frac{-i}{q^2\left(1-\Pi(q^2)\right)}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right).$$

figure 15: the result of the above digram



$$\text{or} \qquad \cdots \frac{e^2 g_{\mu\nu}}{q^2} \cdots \longrightarrow \cdots \frac{Z_3\, e^2 g_{\mu\nu}}{q^2} \cdots .$$

figure 16: the charge regulization

after a long journey of calculation using feymann parameter, wick rotation ,one can work out:

$$i\Pi_2^{\mu\nu}(q^2) = -4ie^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{\frac{1}{2} l_E^2 g^{\mu\nu} - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)}{(l_E^2 + \Delta)^2}$$

with:

$$\Delta = m^2 - x(1-x)q^2$$

which is baddly ultraviolet divergent. and voilate the ward identity.

$\Re$.6 Dimennsional Regularization

The area of a d dimensional unit sphere is:

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

and one can work out the d-dimensional integration using gamma and beta function:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} (\frac{1}{\Delta})^{2 - \frac{d}{2}}$$

then we take the limit $d \to 4$ since:

$$\Gamma(2 - \frac{d}{2}) = \Gamma(2 - \frac{4 - \epsilon}{2}) = \Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$$

then the above integration is:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^2} =_{d\to 4}= \frac{1}{(4\pi)^2}(\frac{2}{\epsilon} - \ln\Delta - \gamma + \ln(4\pi) + O(\epsilon))$$

similarly:

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}(\frac{1}{\Delta})^{n-\frac{d}{2}}$$

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{d}{2}\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}(\frac{1}{\Delta})^{n-\frac{d}{2}-1}$$

in d dimensional space:

$$g^{\mu\nu}g_{\nu\mu} = d$$

$$l^\mu l^\nu - \rightarrow \frac{1}{d}l^2 g^{\mu\nu}$$

the dirac metric becomes a set of d metrics:

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}, tr[1] = 4$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(2-\epsilon)\gamma^\nu$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - \epsilon\gamma^\nu\gamma^\rho$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma$$

after a long journey of calculation one can work out:

$$i\Pi_2^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)i\Pi_2(q^2)$$

with,

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{\frac{d}{2}}}\int_0^1 dxx(1-x)\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}$$

$$\Pi_2(q^2) \sim_{d\rightarrow 4} = -\frac{2\alpha}{\pi}\int_0^1 dxx(1-x)(\frac{2}{\epsilon} - \ln\Delta - \gamma + \ln(4\pi) + O(\epsilon))$$

$$V(r) = -\frac{\alpha}{r}(1 + \frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2mr}}{(mr)^{\frac{3}{2}}} + \cdots)$$

the rediative correction term is called Uehling Potential.

ϒ.8 Functional Method

ℜ.1 Path Integrate in Quantum Mechanics

$$U(q_a, q_b, T) = \langle q_a|e^{-iHT}|q_b\rangle$$

when the hamitonian is weyl ordered we can express it of the form of integration in phase space:

$$U(q_a, q_b, T) = (\prod_i \int Dq(t)Dp(t))exp(i\int_0^T dt\sum_i(p_i\dot{q}_i - H(q_i, p_i)))$$

for a scalor field:

$$U(q_a, q_b, T) = \int D\phi exp(i\int_0^T d^4x\mathcal{L}(\phi(x))$$

ℜ.2 functional quantization of scalor field

The two-point correlation function is expressed as:

$$\langle\Omega|T\{\phi_H(x_1)\phi_H(x_2)\}|\Omega\rangle = \frac{\int D\phi\phi(x_1)\phi(x_2)exp(i\int_{-T}^{T} d^4x\mathcal{L}(\phi(x)))}{\int D\phi exp(i\int_{-T}^{T} d^4x\mathcal{L}(\phi(x)))}$$

we can define a generating functional Z[J]:

$$Z[J] = \int D\phi e^{i\int d^4x(\mathcal{L}+J(x)\phi(x))}$$

one can work out this for free K-G theory:

$$Z[J] = Z_0 e^{-\frac{1}{2}}\int d^4xd^4yJ(x)D_F(x-y)J(y)$$

then we can use this generating functional to get the correlation functions:

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|\Omega\rangle = \frac{1}{Z[J]}(-i\frac{\delta}{\delta J(x_1)})(-i\frac{\delta}{\delta J(x_2)})\cdots(-i\frac{\delta}{\delta J(x_n)})Z[J]|_J = 0$$

Z[J] is like the portion function in statistical physics,and J(x) is like the external field.

ℜ.3 functional quantization of EM field

similarly,using the Faddeev and Popov trick,one can work out the correlation functions for EM Field:

$$\langle\Omega|TO(A)|\Omega\rangle = lim_{T\to\infty(1-i\epsilon)}\frac{\int DAO(A)e^{i\int_{-T}^{T} d^4x[\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2]}}{\int DAe^{i\int_{-T}^{T} d^4x[\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2]}}$$

and the proton propogator is:

$$\frac{-i}{k^2+i\epsilon}(g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2})$$

ℜ.4 functional quantization of Dirac field

Υ.9 Remarks On The Normal Operator

很多时候对于 Normal Operator N 有很多疑惑。比如：

$$N(aa^\dagger - a^\dagger a) = a^\dagger a - a^\dagger a = 0$$

但是，从另外一个角度出发：

$$N(aa^\dagger - a^\dagger a) = N([a, a^\dagger]) = N(1) = 1$$

这就导出了矛盾，这说明 N 的定义还不是很完备，以下详细讨论一下 Normal Operator N

ℜ.1 Normal Operator 的作用空间

简单来讲，Normal Operator 是作用在自由幺半群代数空间上的算符。比如，当我们考虑玻色体系，描述这个体系的产生湮灭算符集为：

$$\{a_p, a_p^\dagger\}_p$$

利用这个算符集可以自由生成一个幺半群 G，其中这个幺半群 G 中的元素为：

$$\prod_p x_p$$

其中 $x_p = a_p$ 或者 $x_p = a_p^\dagger$。这个半群的乘法为两个元素的连写:

$$\prod_p x_p \circ \prod_{p'} x_{p'} = \prod_p x_p \prod_{p'} x_{p'}$$

空集为此幺半群的单位元，记为 e。利用这个幺半群 G 构造 G 上的 C-代数空间 C[G], 其中 C[G] 上的元素为:

$$f = \sum_{g \in G} c_g g$$

其中元素的相等定义为:

$$\sum_g c_g g = \sum_g d_g g \Leftrightarrow c_g = d_g, \forall g$$

其中加法定义为：

$$\sum_g c_g g + \sum_g d_g g = \sum_g (c_g + d_g) g$$

数乘定义为:

$$c_1 \sum_g c_g g = \sum_g c_1 c_g g$$

乘法定义为：

$$\sum_g c_g g \sum_{g'} c_{g'} g' = \sum_g \sum_{g'} c_g c_{g'} g g'$$

Normal Opertor N 的作用空间就是上述定义的群代数空间。

ℜ.2 Normal Operator N 的定义

定义在上述群代数空间上的满足以下几个性质的算符称为 Normal Operator，记为 N:

- N 是作用在群代数空间上的线性算符:

$$N(f+g) = N(f) + N(g), \forall f, g \in C[G]$$

- N 在元素 e 上的作用不变

$$N(e) = e$$

因此对于任意一个 $c \in C$

$$N(c \circ e) = c \circ e$$

- 产生算符在左边时，可以直接拿出来

$$N(a_p^\dagger f) = a_p^\dagger N(g), \forall f \in C[G]$$

- 湮灭算符在右边时也可以拿出来

$$N(f a_p) = N(f) a_p, \forall f \in C[G]$$

- N 算符的作用是把产生算符符号放到左侧

$$N(f a_p^\dagger g) = a_p^\dagger N(fg), \forall f, g \in C[G]$$

ℜ.3 具体描述

由于 N 的作用空间是一个自由半群的 C 代数空间，因此对于任意一个物理希尔伯特空间上的算符，在进行 N 操作时，必须将其映射到 C[G] 上，然后再利用相同的映射将其转化为物理上的算符。也就是说，放在 N() 里的东西已经自然的是 C[G] 上的元素。为了区别这两个空间中的元素，我们假设有一个真实的物理系统，描述该系统的产生湮灭算符集为：

$$\{a_p, a_p^\dagger\}$$

这些物理上的算符成立等式：

$$a_p a_{p'}^\dagger - a_{p'}^\dagger a_p = (2\pi)^3 \delta^{(3)}(p - p')$$

任何物理上的算符都是这个物理算符集上构造出来的群代数空间 C[G] 上的元素，但是元素之间有由上述对易关系给出的某些关系，比如上面的对易关系就是群代数空间 C[G] 上的两个元素，但是这两个元素相等 (物理上)，因此这个空间实际上并不是一个一个自洽的群代数空间，因为：

$$\sum_g c_g g = \sum_g d_g g \Leftrightarrow c_g = d_g, \forall g$$

为此，在进行 N 操作之前，我们需要构造出一个自洽的自由群代数空间。我们通过一个一一映射 F_{set}(指标 set 表示这是算符集与符号集之间的映射) 将此物理算符集映射到某个符号集，如：

$$F_{set}(a_p) = \boxtimes_p, F_{set}(a_p^\dagger) = \bigcirc_p$$

这里利用符号 $\boxtimes_p, \bigcirc_p$ 是为了说明映射过去用于构造自洽群代数空间的元素集与之前的物理算符之间没有任何关系。那么这时候得到一个符号集合

$$\Lambda = \{\boxtimes_p, \bigcirc_p\}$$

利用此符号集构造一个自洽的自由幺半群代数空间 $C[G_2]$，这个空间是自洽的群代数空间是因为：

$$\boxtimes_p \bigcirc_{p'} - \bigcirc_{p'} \boxtimes_p \neq (2\pi)^3 \delta^{(3)}(p - p') \circ e$$

那么在空间 $C[G_2]$ 上可以定义 N 算符。物理上常用的 N_{phy} 算符 (为区分，我们将其标记为 N_{phy}) 实际上是以下三个映射的复合:

$$N_{phy} = F^{-1} \circ N \circ F$$

其中 F 为前述的 F_{set} 的自然延拓，使得 F 成为由物理算符集构造的非自洽的自由群代数 C[G] 到 C[G_2] 的一个同构。

ℜ.4 一些例子

比如考虑物理上的单模光场，描述该系统的物理算符集为

$$\{a, a^\dagger\}$$

考虑算符

$$h(a, a^\dagger) = aa^\dagger - a^\dagger a$$

那么我们有

$$N_{phy}(h(a, a^\dagger)) = F^{-1} \circ N \circ F(h(a, a^\dagger))$$

首先有

$$F(h(a, a^\dagger)) = \boxtimes \bigcirc - \bigcirc \boxtimes$$

然后我们有:

$$N \circ F(h(a, a^\dagger)) = N(\boxtimes \bigcirc - \bigcirc \boxtimes) = \bigcirc \boxtimes - \bigcirc \boxtimes = 0 \circ e$$

从而我们有:

$$N_{phy}(h(a, a^\dagger)) = F^{-1}(0 \circ e) = 0 \bullet 1 = 0$$

同理

$$N_{phy}(1) = F^{-1} \circ N(e) = F^{-1}(e) = 1$$

但是

$$N_{phy}(1) \neq N_{phy}(aa^\dagger - a^\dagger a)$$

这是因为

$$F(1) = e \neq \boxtimes \bigcirc - \bigcirc \boxtimes = F(aa^\dagger - a^\dagger a)$$

ℜ.5 结论

物理上常常采用相同的一套符号，即:

$$F_{set}(a_p) = a_p$$

$$F_{set}(a_p^\dagger) = a_p^\dagger$$

$$N_{phy} = N$$

因此常常引起各种不自洽的等式，例如本文开始部分引入的矛盾。只要搞清楚了本文给出的这些概念那么这些矛盾立刻被消除了。简单来讲，放在 N 算符里的东西只是一个符号，不能进行任何运算。类似地，可以考虑费米子系统。由于核心思想是一致的，因此就略去了。